

# A FAMILY OF CLASS-2 NILPOTENT GROUPS, THEIR AUTOMORPHISMS AND PRO-ISOMORPHIC ZETA FUNCTIONS

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**ABSTRACT.** The pro-isomorphic zeta function  $\zeta_\Gamma^\wedge(s)$  of a finitely generated nilpotent group  $\Gamma$  is a Dirichlet generating function that enumerates finite-index subgroups whose profinite completion is isomorphic to that of  $\Gamma$ . Such zeta functions can be expressed as Euler products of  $p$ -adic integrals over the  $\mathbb{Q}_p$ -points of an algebraic automorphism group associated to  $\Gamma$ . In this way they are closely related to classical zeta functions of algebraic groups over local fields.

We describe the algebraic automorphism groups for a natural family of class-2 nilpotent groups  $\Delta_{m,n}$  of Hirsch length  $\binom{m+n-2}{n-1} + \binom{m+n-1}{n-1} + n$  and central Hirsch length  $n$ ; these groups can be viewed as generalisations of  $D^*$ -groups of odd Hirsch length. General  $D^*$ -groups, that is ‘indecomposable’ finitely generated, torsion-free class-2 nilpotent groups with central Hirsch length 2, were classified up to commensurability by Grunewald and Segal.

We calculate the local pro-isomorphic zeta functions for the groups  $\Delta_{m,n}$  and obtain, in particular, explicit formulae for the local pro-isomorphic zeta functions associated to  $D^*$ -groups of odd Hirsch length. From these we deduce local functional equations; for the global zeta functions we describe the abscissae of convergence and find meromorphic continuations. We deduce that the spectrum of abscissae of convergence for pro-isomorphic zeta functions of class-2 nilpotent groups contains infinitely many cluster points. For instance, the global abscissa of convergence of the pro-isomorphic zeta function of a  $D^*$ -group of Hirsch length  $2m + 3$  is shown to be  $6 - \frac{15}{m+3}$ .

## 1. INTRODUCTION

Zeta functions provide a compact and powerful way to encode information about the lattice of finite-index subgroups of a finitely generated group. Our attention focuses on finitely generated, torsion-free nilpotent groups, for which a rich theory is available; see [7] and references therein. In the current paper we are interested in the *pro-isomorphic zeta function* of such a group  $\Gamma$ , i.e., the Dirichlet generating function

$$\zeta_\Gamma^\wedge(s) = \sum_{n=1}^{\infty} \frac{a_n^\wedge(\Gamma)}{n^s},$$

where  $a_n^\wedge(\Gamma)$  denotes the number of subgroups  $\Delta$  of index  $n$  in  $\Gamma$  such that the profinite completion  $\widehat{\Delta}$  is isomorphic to the profinite completion  $\widehat{\Gamma}$  of the ambient group.

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It follows from the nilpotency of  $\Gamma$  that one obtains an Euler product decomposition over all rational primes  $p$ :

$$(1.1) \quad \zeta_{\Gamma}^{\wedge}(s) = \prod_p \zeta_{\Gamma,p}^{\wedge}(s), \quad \text{where} \quad \zeta_{\Gamma,p}^{\wedge}(s) = \sum_{k=0}^{\infty} a_{p^k}^{\wedge}(\Gamma) p^{-ks}$$

is called the *local zeta function* at a prime  $p$ ; see [9]. Furthermore, each of the local factors  $\zeta_{\Gamma,p}^{\wedge}(s)$  is a rational function over  $\mathbb{Q}$  in  $p^{-s}$ .

In comparison to other zeta functions of groups, a unique feature of pro-isomorphic zeta functions  $\zeta_{\Gamma}^{\wedge}(s)$  is their relation to a rather different object of independent interest. Let  $\mathcal{K}$  be a number field with ring of integers  $\mathcal{O}$ , and let  $\mathbf{G} \subseteq \mathbf{GL}_d$  be an affine group scheme over  $\mathcal{O}$  with a fixed embedding into  $\mathbf{GL}_d$ . For a finite prime  $\mathfrak{p}$ , let  $\mathcal{K}_{\mathfrak{p}}$  denote the completion at  $\mathfrak{p}$ , and let  $\mathcal{O}_{\mathfrak{p}}$  denote the valuation ring of  $\mathcal{K}_{\mathfrak{p}}$ . Putting  $\mathbf{G}_{\mathfrak{p}} = \mathbf{G}(\mathcal{K}_{\mathfrak{p}})$ ,  $\mathbf{G}_{\mathfrak{p}}^+ = \mathbf{G}(\mathcal{K}_{\mathfrak{p}}) \cap \text{Mat}_d(\mathcal{O}_{\mathfrak{p}})$  and  $\mathbf{G}(\mathcal{O}_{\mathfrak{p}}) = \mathbf{G}(\mathcal{K}_{\mathfrak{p}}) \cap \mathbf{GL}_d(\mathcal{O}_{\mathfrak{p}})$ , one defines the zeta function of  $\mathbf{G}$  at  $\mathfrak{p}$  as

$$(1.2) \quad \mathcal{Z}_{\mathbf{G},\mathfrak{p}}(s) = \int_{\mathbf{G}_{\mathfrak{p}}^+} |\det(g)|_{\mathfrak{p}}^s d\mu_{\mathbf{G}_{\mathfrak{p}}}(g),$$

where  $|\cdot|_{\mathfrak{p}}$  is the  $\mathfrak{p}$ -adic absolute value on  $\mathcal{K}_{\mathfrak{p}}$  and  $\mu_{\mathbf{G}_{\mathfrak{p}}}$  is the right Haar measure on  $\mathbf{G}(k_{\mathfrak{p}})$ , normalised so that  $\mu_{\mathbf{G}_{\mathfrak{p}}}(\mathbf{G}(\mathcal{O}_{\mathfrak{p}})) = 1$ . In the past century,  $\mathcal{Z}_{\mathbf{G},\mathfrak{p}}(s)$  was studied by Hey, Weil, Tamagawa, Satake, Macdonald and Igusa [10, 18, 17, 16, 15, 12], for independent reasons. Grunewald, Segal and Smith [9] recognised that these classical zeta functions of algebraic groups relate to pro-isomorphic zeta functions of nilpotent groups; see Section 1.2. Studying the pro-isomorphic zeta function of a finitely generated nilpotent group typically involves two steps: first one needs to understand an associated algebraic automorphism group  $\mathbf{G}$  which comprises an affine group scheme over  $\mathbb{Z}$ , and subsequently one studies the  $\mathfrak{p}$ -adic integral (1.2).

**1.1. Main results.** In this paper we consider a certain family of finitely generated, torsion-free class-2 nilpotent groups  $\Delta_{m,n}$ , defined below, and set about explicitly computing the associated algebraic automorphism groups and local pro-isomorphic zeta functions.

Our original interest concerned finitely generated, torsion-free class-2 nilpotent groups of central Hirsch length 2; we refer to such groups as  $D^*$ -groups. Up to commensurability,  $D^*$ -groups were classified by Grunewald and Segal [8] in terms of indecomposable constituents of their radicable hulls; the latter are called  $\mathcal{D}^*$ -groups and our terminology is a natural adaptation. For any integer  $m \geq 1$ , there is a unique commensurability class of ‘indecomposable’  $D^*$ -groups of Hirsch length  $2m + 3$ , represented by the group

$$\Gamma_m = \langle a_1, \dots, a_m, b_1, \dots, b_{m+1}, c_1, c_2 \mid R \rangle,$$

where the finite set of relations  $R$  specifies that  $c_1, c_2$  are central and

$$[a_i, a_j] = 1, \quad [b_i, b_j] = 1, \quad [a_i, b_j] = \begin{cases} c_1 & \text{if } i = j, \\ c_2 & \text{if } i + 1 = j, \\ 1 & \text{otherwise,} \end{cases} \quad \text{for all relevant indices.}$$

For these  $D^*$ -groups of odd Hirsch length we obtain the following results.

**Theorem 1.1.** *Let  $\Gamma_m$  be the ‘indecomposable’  $D^*$ -group of Hirsch length  $2m+3$  defined above, for  $m \geq 1$ . Then, for every rational prime  $p$ ,*

$$\zeta_{\Gamma_m, p}^\wedge(s) = \frac{1 + p^{\frac{1}{2}(9m^2+m-2)-(m^2+2m-1)s}}{(1 - p^{\frac{1}{2}m(9m+1)-(m^2+2m-1)s})(1 - p^{4m+2-(m+2)s})(1 - p^{6m+2-(m+3)s})}.$$

**Remark 1.2.** We remark that the ‘indecomposable’  $D^*$ -group  $\Gamma_1$  of Hirsch length 5 is none other than the Grenham group of the same Hirsch length whose local pro-isomorphic zeta functions have previously been determined: indeed, setting  $m = 1$  in Theorem 1.1 or setting  $d = 3$  in the formula appearing in [1, Section 3.3.13.2], we obtain the same expression

$$\zeta_{\Gamma_1, p}^\wedge(s) = \frac{1}{(1 - p^{4-2s})(1 - p^{5-2s})(1 - p^{6-3s})}.$$

**Corollary 1.3.** *Let  $\Gamma_m$  be the ‘indecomposable’  $D^*$ -group of Hirsch length  $2m+3$  defined above, for  $m \geq 1$ . Then the pro-isomorphic zeta function  $\zeta_{\Gamma_m}^\wedge(s)$  satisfies local functional equations and admits meromorphic continuation to the entire complex plane. It has abscissa of convergence 3, if  $m = 1$ , and  $6 - \frac{15}{m+3}$ , if  $m \geq 2$ .*

The occurrence of local functional equations for pro-isomorphic zeta functions is a widespread but not ubiquitous phenomenon; cf. [3]. In the present instance it refers to the fact that

$$\zeta_{\Gamma_m, p}^\wedge(s)|_{p \rightarrow p^{-1}} = -p^{10m+5-(2m+5)s} \zeta_{\Gamma_m, p}^\wedge(s) \quad \text{for every rational prime } p,$$

using the formula in Theorem 1.1. The theorem and its corollary significantly widen the scope of known examples. In particular, they indicate that there is a rich spectrum of abscissae of convergence for pro-isomorphic zeta functions of class-2 nilpotent groups; cf. [5, Problem 1.3].

In fact, we study a much larger family of class-2 nilpotent groups, generalising  $D^*$ -groups of odd Hirsch length in the following way. Let  $m, n \in \mathbb{N}$  with  $n \geq 2$ . Put

$$\begin{aligned} \mathbf{E} &= \{\mathbf{e} \mid \mathbf{e} = (e_1, \dots, e_n) \in \mathbb{N}_0^n \text{ with } e_1 + \dots + e_n = m-1\}, \\ \mathbf{F} &= \{\mathbf{f} \mid \mathbf{f} = (f_1, \dots, f_n) \in \mathbb{N}_0^n \text{ with } f_1 + \dots + f_n = m\}. \end{aligned}$$

We consider the group  $\Delta_{m, n}$  on  $|\mathbf{E}| + |\mathbf{F}| + n$  generators

$$\{a_{\mathbf{e}} \mid \mathbf{e} \in \mathbf{E}\} \cup \{b_{\mathbf{f}} \mid \mathbf{f} \in \mathbf{F}\} \cup \{c_j \mid j \in \{1, \dots, n\}\},$$

subject to the defining relations

$$[a_{\mathbf{e}}, a_{\mathbf{e}'}] = [b_{\mathbf{f}}, b_{\mathbf{f}'}] = [a_{\mathbf{e}}, c_j] = [b_{\mathbf{f}}, c_j] = [c_j, c_{j'}] = 1$$

for all  $\mathbf{e}, \mathbf{e}' \in \mathbf{E}$ ,  $\mathbf{f}, \mathbf{f}' \in \mathbf{F}$ ,  $j, j' \in \{1, \dots, n\}$  and

$$[a_{\mathbf{e}}, b_{\mathbf{f}}] = \begin{cases} c_i & \text{if } \mathbf{f} - \mathbf{e} \text{ is of the form } (\underbrace{0, \dots, 0}_{i-1 \text{ entries}}, 1, \underbrace{0, \dots, 0}_{n-i \text{ entries}}) \text{ for some } i, \\ 1 & \text{if } \mathbf{f} - \mathbf{e} \text{ is not of this form,} \end{cases}$$

for  $\mathbf{e} \in \mathbf{E}$  and  $\mathbf{f} \in \mathbf{F}$ . Consequently,  $\Delta_{m,n}$  is a finitely generated, torsion-free class-2 nilpotent group of Hirsch length

$$|\mathbf{E}| + |\mathbf{F}| + n = \binom{m+n-2}{n-1} + \binom{m+n-1}{n-1} + n$$

whose centre coincides with the commutator subgroup and has Hirsch length  $n$ ; indeed,

$$Z(\Delta_{m,n}) = [\Delta_{m,n}, \Delta_{m,n}] = \langle c_1, \dots, c_n \rangle.$$

We can now state our main result. For the subsequent analysis, it is convenient to think of  $m$  as a primary parameter, each  $m$  yielding a sequence of groups  $\Delta_{m,n}$  for  $n \in \mathbb{N}_{\geq 2}$ ; we choose to write binomial coefficients in accordance with this.

**Theorem 1.4.** *Let  $\Delta_{m,n}$  be the nilpotent group defined above, for  $m \geq 1$  and  $n \geq 2$ . Let  $\Phi$  be the set of roots of the algebraic group  $\mathbf{GL}_n$ , with negative roots  $\Phi^-$  determined by some choice of simple roots  $\alpha_1, \dots, \alpha_{n-1}$ , and let  $\ell$  denote the standard length function on the Weyl group  $W \cong \text{Sym}(n)$  acting on  $\Phi$ . Then, for all rational primes  $p$ ,*

$$(1.3) \quad \zeta_{\Delta_{m,n},p}^\wedge(s) = \frac{\sum_{w \in W} p^{-\ell(w)} \prod_{i=1}^{n-1} X_i^{\nu_i(w)}}{\prod_{i=1}^{n-1} (1 - X_i)} \cdot \frac{1}{(1 - \tilde{X}_0)(1 - \tilde{X}_n)},$$

where

$$\nu_i(w) = \begin{cases} 1 & \text{if } \alpha_i \in w(\Phi^-), \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq i \leq n-1,$$

$X_i = p^{A_i - B_i s}$ , for  $1 \leq i \leq n-1$ , and  $\tilde{X}_j = p^{\tilde{A}_j - \tilde{B}_j s}$ , for  $j \in \{0, n\}$ , with

$$\begin{aligned} A_i &= i(n-i) + \left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right) ((m-1)n + i) \\ &\quad + \sum_{j=1}^i \left( 1 + \frac{(m-1)(i-j+1)}{n-j+1} \right) \binom{m+j-2}{m-1} \binom{m+n-j-1}{m-1}, \\ B_i &= -m(m-1) \binom{m+n-2}{m} + \left( 1 + \binom{m+n-2}{m-1} \right) ((m-1)n + i) \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_0 &= n \left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right), & \tilde{A}_n &= \tilde{A}_0 + \binom{2m+n-2}{2m-1}, \\ \tilde{B}_0 &= \binom{m+n-2}{m-1} + n, & \tilde{B}_n &= \tilde{B}_0 + \binom{m+n-2}{m} = \binom{m+n-1}{m} + n. \end{aligned}$$

Furthermore, these parameters are connected by the relations

$$(m-1)\tilde{A}_0 = A_0, \quad (m-1)\tilde{B}_0 = B_0, \quad m\tilde{A}_n = A_n, \quad m\tilde{B}_n = B_n.$$

We remark that the group  $\Delta_{m,2}$  is isomorphic to the  $D^*$ -group  $\Gamma_m$  discussed earlier and indeed, Theorem 1.1 is obtained as a special case of Theorem 1.4 by setting  $n = 2$ . The following can be deduced by a method of Igusa [12] using symmetries of the Weyl group  $W$ ; further details are provided in Section 5.

**Corollary 1.5.** *Let  $\Delta_{m,n}$  be the nilpotent group defined above, for  $m \geq 1$  and  $n \geq 2$ . Then for all primes  $p$ , the local zeta function  $\zeta_{\Delta_{m,n},p}^\wedge(s)$  satisfies a functional equation of the form*

$$\zeta_{\Delta_{m,n},p}^\wedge(s)|_{p \rightarrow p^{-1}} = (-1)^{n-1} p^{a+bs} \zeta_{\Delta_{m,n},p}^\wedge(s)$$

where

$$(1.4) \quad \begin{aligned} a &= \binom{n}{2} + 2n \left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right) + \binom{2m+n-2}{2m-1}, \\ b &= (2m-1) \binom{m+n-2}{m} - 2n \left( 1 + \binom{m+n-2}{m-1} \right). \end{aligned}$$

This reduces to the functional equation for  $\zeta_{\Gamma_{m,p}}^{\wedge}(s)$  by setting  $n = 2$ . A slight modification of the main result in [2] can be used to deduce functional equations already half way through the proof of our theorem; see Remark 3.3. A thorough analysis of the explicit formulae in Theorem 1.4 yields the following information about convergence and meromorphic continuation; see Section 5.

**Corollary 1.6.** *Let  $\Delta_{m,n}$  be the nilpotent group defined above, for  $m \geq 1$  and  $n \geq 2$ . Then the pro-isomorphic zeta function  $\zeta_{\Delta_{m,n}}^{\wedge}(s)$  has abscissa of convergence*

$$\alpha(m, n) = \begin{cases} (A_1 + 1)/B_1 = n + 1 & \text{if } m = 1, \\ (\tilde{A}_0 + 1)/\tilde{B}_0 & \text{if } (m, n) \in \mathcal{C}, \\ (\tilde{A}_n + 1)/\tilde{B}_n & \text{otherwise,} \end{cases}$$

where  $\tilde{A}_j, \tilde{B}_j$  are as in Theorem 1.4 and

$$\mathcal{C} = (\{2, 3\} \times \mathbb{N}_{\geq 3}) \cup (\{4\} \times \{4, 5, \dots, 38\}) \cup (\{5\} \times \{5, 6, 7, 8, 9\}).$$

Moreover,  $\zeta_{\Delta_{m,n}}^{\wedge}(s)$  admits meromorphic continuation (at least) to the half-plane consisting of all  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > \beta(m, n)$ , where

$$\beta(m, n) = \max\{A_i/B_i \mid 1 \leq i \leq n-1\} < \alpha(m, n).$$

The resulting pole of  $\zeta_{\Delta_{m,n}}^{\wedge}(s)$  at  $s = \alpha(m, n)$  is simple.

**Remark 1.7.** We remark that the group  $\Delta_{1,n}$  of Hirsch length  $2n+1$  is none other than the Grenham group of the same Hirsch length, whose local pro-isomorphic zeta functions have previously been determined by the first author. Setting  $m = 1$  in Theorem 1.4 or setting  $d = n+1$  in the formula appearing in [1, Section 3.3.13.2] we obtain different but equivalent expressions for the local pro-isomorphic zeta functions of  $\Delta_{1,n}$ , namely

$$(1.5) \quad \zeta_{\Delta_{1,n,p}}^{\wedge}(s) = \frac{\sum_{w \in W} p^{-\ell(w)} \prod_{i=1}^{n-1} (p^{i(2n+2-i)-2is})^{\nu_i(w)}}{(1 - p^{n(n+1)-(n+1)s}) \prod_{i=1}^n (1 - p^{i(2n+2-i)-2is})},$$

where  $\ell$  denotes the standard length function on  $W \cong \operatorname{Sym}(n)$ , and

$$(1.6) \quad \zeta_{\Delta_{1,n,p}}^{\wedge}(s) = \frac{1}{(1 - p^{n(n+1)-(n+1)s}) \prod_{i=1}^n (1 - p^{n+1+i-2s})}.$$

The formula (1.6) can be derived by a direct computation that does not require the general approach via Bruhat decompositions; compare [1, Section 3.3.2]. The connection between (1.5) and (1.6) is given by the identity

$$\frac{\sum_{w \in W} p^{-\ell(w)} \prod_{i=1}^{n-1} (p^{i(n-i)-it})^{\nu_i(w)}}{\prod_{i=1}^n (1 - p^{i(n-i)-it})} = \int_{\operatorname{GL}_n(\mathbb{Q}_p)^+} |\det B|_p^t d\mu = \frac{1}{\prod_{i=1}^n (1 - p^{i-1-t})}$$

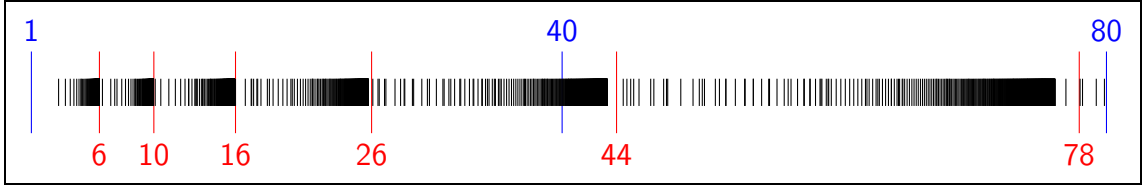
applied to  $t = 2s - n - 2$ . Observe that by virtue of (1.6), the zeta function  $\zeta_{\Delta_{1,n,p}}^{\wedge}(s)$  admits meromorphic continuation to the entire complex plane.

It would be interesting to study further analytic properties of the global zeta functions associated to the nilpotent groups  $\Delta_{m,n}$ . Moreover it would be illuminating to study in more depth the distribution of the abscissae of convergence  $\alpha(m, n)$  of the groups  $\Delta_{m,n}$ ; compare Figure 1. A straightforward analysis yields the following corollary, which shows that the spectrum of abscissae of convergence for pro-isomorphic zeta functions of class-2 nilpotent groups contains infinitely many cluster points; cf. [5, Problem 1.3].

**Corollary 1.8.** *For each  $n \geq 2$ , the abscissae  $\alpha(m, n)$  of  $\Delta_{m,n}$  converge, as  $m \rightarrow \infty$ , namely to*

$$\lim_{m \rightarrow \infty} \alpha(m, n) = 2n + 2^{n-1}.$$

FIGURE 1. Abscissae of convergence  $\alpha(m, n)$  within the real interval  $[0, 80]$  arising from parameters  $2 \leq m \leq 500$  and  $2 \leq n \leq 20$



The first six cluster points provided by Corollary 1.8, the numbers 6, 10, 16, 26, 44, 78 are indicated in red. One can observe that the rate of convergence is fairly slow.

**1.2. Methods.** Finally we give a brief indication how pro-isomorphic zeta functions of nilpotent groups relate to zeta functions of algebraic groups. This gives us the opportunity to discuss the methods used and the relevance of our specific results in the context of the general theory.

To a finitely generated, torsion-free nilpotent group  $\Gamma$  one associates, via Lie theory, a  $\mathbb{Z}$ -Lie lattice  $L$  of finite rank, whose local zeta functions  $\zeta_{L_p}^\wedge(s) = \sum_{k=0}^\infty b_{p^k}^\wedge(L_p) p^{-ks}$  satisfy  $\zeta_{\Gamma,p}^\wedge(s) = \zeta_{L_p}^\wedge(s)$  for almost all rational primes  $p$ . Here  $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$  denotes the  $p$ -adic completion of  $L$ , and  $b_{p^k}^\wedge(L_p)$  is the number of Lie sublattices of  $L_p$  of index  $p^k$  which are isomorphic to  $L_p$ , or equivalently, are the image of  $L_p$  under a Lie endomorphism of  $\mathbb{Q}_p \otimes L$ . Next recall the notion of the *algebraic automorphism group* of  $L$ : this group  $G = \text{Aut}(L)$  is realised, via a  $\mathbb{Z}$ -basis of  $L$ , as an affine group scheme  $G \subseteq \text{GL}_d$  over  $\mathbb{Z}$ , where  $d$  denotes the  $\mathbb{Z}$ -rank of  $L$ , so that  $\text{Aut}_K(K \otimes_{\mathbb{Z}} L) \cong G(K) \subseteq \text{GL}_d(K)$  for every extension field  $K$  of  $\mathbb{Q}$ . With the given arithmetic structure, we have  $\text{Aut}(L) \cong G(\mathbb{Z})$  and  $\text{Aut}(L_p) \cong G(\mathbb{Z}_p)$  for every rational prime  $p$ . In [9], Grunewald, Segal and Smith showed that

$$\zeta_{\Gamma,p}^\wedge(s) = \zeta_{L_p}^\wedge(s) = \mathcal{Z}_{G,p}(s),$$

where  $\mathcal{Z}_{G,p}(s)$  is a special instance of the  $\mathfrak{p}$ -adic integral (1.2); the first equality holds for almost all primes and the second for all primes  $p$ .

An explicit finite form for  $\mathcal{Z}_{G,p}(s)$ , subject to technical restrictions on  $G \subseteq \text{GL}_d$ , was obtained by Igusa [12], subsequently generalised by du Sautoy and Lubotzky [6] and by the first author [2]. The essential idea behind these results is to reduce the

integral (1.2) to an integral over a reductive  $\mathfrak{p}$ -adic group  $H_{\mathfrak{p}}$  and then apply a  $\mathfrak{p}$ -adic Bruhat decomposition established by Iwahori and Matsumoto [13]. Upon further combinatorial analysis one obtains a weighted sum of generating functions over cones, indexed by elements of the Weyl group of  $H$ .

The reduction of the integral (1.2) to an integral over a connected reductive group, due to du Sautoy and Lubotzky, can be summarised as follows.

**Theorem 1.9** (cf. [6, Theorem 2.2]). *Under various assumptions on  $G$  and  $\mathfrak{p}$ , which are satisfied for almost all primes  $\mathfrak{p}$ ,*

$$\mathcal{Z}_{G,\mathfrak{p}}(s) = \int_{H_{\mathfrak{p}}^+} |\det(h)|_{\mathfrak{p}}^s \vartheta(h) d\mu_{H_{\mathfrak{p}}}(h),$$

where  $H$  is a connected reductive complement in  $G^\circ$  of the unipotent radical  $N$  of  $G$  and  $\vartheta(h) = \mu_{N_{\mathfrak{p}}}(\{n \in N_{\mathfrak{p}} \mid nh \in G_{\mathfrak{p}}^+\})$  for each  $h \in H_{\mathfrak{p}}^+$ .

Despite an encouraging range of valuable insights, our understanding of pro-isomorphic zeta functions of finitely generated nilpotent groups is far from complete. For instance, in [2], the first author generalised earlier results of Igusa [12] and du Sautoy and Lubotzky [6] showing that the pro-isomorphic zeta functions of split algebraic groups satisfy – under suitable technical conditions – local functional equations. However, it is currently not possible to effectively predict whether the technical assumptions involved hold for the algebraic automorphism group associated to a finitely generated nilpotent group without actually determining the automorphism groups in full. Moreover, the algebraic automorphism groups that have been described explicitly display only a comparatively small degree of complexity, especially regarding the unipotent radical.

It is a natural task to reveal more variety in the features of pro-isomorphic zeta functions by studying new families of groups where we continue to have some level of control. In the current paper we treat the finitely generated, torsion-free class-2 nilpotent groups  $\Delta_{m,n}$  and obtain further evidence for the existence of local functional equations for pro-isomorphic zeta functions in nilpotency class 2.

The function  $\vartheta(h)$  that features in Theorem 1.9 is perhaps the least understood ingredient in the study of pro-isomorphic zeta functions of nilpotent groups. In [6, Theorem 2.3], du Sautoy and Lubotzky state that, if the  $\mathfrak{p}$ -adic group  $H_{\mathfrak{p}}$  arises from the reductive part  $H$  of the algebraic automorphism group associated to an arbitrary class-2 nilpotent group, then the function  $\vartheta(h)$  is a character on  $H_{\mathfrak{p}}$ . In the current paper, we find counterexamples to this statement. Indeed, for the algebraic automorphism groups associated to  $\Delta_{m,n}$  we find that  $\vartheta(h)$  is obtained in general not from a character but rather from a piecewise-character on a maximal torus in the sense considered in [2, Lemma 3.12], thus giving a realisation of this behaviour of  $\vartheta(h)$  for an infinite family of groups. Previously this phenomenon was known to occur only from one isolated example, namely the group of upper unitriangular  $4 \times 4$  matrices over  $\mathbb{Z}$ .

In our preprint [4], on pro-isomorphic zeta functions of  $D^*$ -groups of even Hirsch length, we describe an example of a  $D^*$ -group for which the function  $\vartheta(h)$  is highly exotic and very far from being a character on  $H_{\mathfrak{p}}$ . Thus even within the comparatively restrictive setting of class-2 nilpotent groups, the picture is much more complex than previously expected and deserves further study.



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## 2. THE LIE LATTICES $L_{m,n}$ AND THEIR ALGEBRAIC AUTOMORPHISM GROUPS

**Definition 2.1.** Let  $m, n \in \mathbb{N}$  with  $n \geq 2$ . Put

$$\begin{aligned}\mathbf{E} &= \{\mathbf{e} \mid \mathbf{e} = (e_1, \dots, e_n) \in \mathbb{N}_0^n \text{ with } e_1 + \dots + e_n = m - 1\}, \\ \mathbf{F} &= \{\mathbf{f} \mid \mathbf{f} = (f_1, \dots, f_n) \in \mathbb{N}_0^n \text{ with } f_1 + \dots + f_n = m\}.\end{aligned}$$

We consider the  $\mathbb{Z}$ -Lie lattice  $L = L_{m,n}$  on the generators

$$(2.1) \quad \{x_{\mathbf{e}} \mid \mathbf{e} \in \mathbf{E}\} \cup \{y_{\mathbf{f}} \mid \mathbf{f} \in \mathbf{F}\} \cup \{z_j \mid j \in \{1, \dots, n\}\},$$

so that  $L$  has  $\mathbb{Z}$ -rank

$$\text{rank}_{\mathbb{Z}}(L) = \binom{m+n-2}{n-1} + \binom{m+n-1}{n-1} + n,$$

subject to the defining relations

$$[x_{\mathbf{e}}, x_{\mathbf{e}'}] = [y_{\mathbf{f}}, y_{\mathbf{f}'}] = [x_{\mathbf{e}}, z_j] = [y_{\mathbf{f}}, z_j] = [z_j, z_{j'}] = 0$$

for all  $\mathbf{e}, \mathbf{e}' \in \mathbf{E}$ ,  $\mathbf{f}, \mathbf{f}' \in \mathbf{F}$ ,  $j, j' \in \{1, \dots, n\}$  and

$$[x_{\mathbf{e}}, y_{\mathbf{f}}] = \begin{cases} z_i & \text{if } \mathbf{f} - \mathbf{e} \text{ is of the form } (\underbrace{0, \dots, 0}_{i-1 \text{ entries}}, 1, \underbrace{0, \dots, 0}_{n-i \text{ entries}}) \text{ for some } i, \\ 0 & \text{if } \mathbf{f} - \mathbf{e} \text{ is not of this form} \end{cases}$$

for  $\mathbf{e} \in \mathbf{E}$  and  $\mathbf{f} \in \mathbf{F}$ . Consequently,  $L$  is nilpotent of class 2, and

$$Z(L) = [L, L] = \langle z_1, \dots, z_n \rangle$$

has  $\mathbb{Z}$ -rank  $n$ .

**Remark 2.2.** We observe that  $L = L_{m,n}$  can be naturally identified with a graded Lie ring associated to the class-2 nilpotent group  $\Delta = \Delta_{m,n}$  introduced in Section 1.1. Indeed, because  $\Delta$  has class 2, it is elementary to set up a naive Lie correspondence as follows:  $\Delta/Z(\Delta) \oplus Z(\Delta)$  carries the structure of a graded Lie ring with respect to the usual Lie bracket operation induced by commutation in  $\Delta$ . Conversely, the Lie ring  $L$  fully determines the group  $\Delta$ . For any rational prime  $p$ , we denote by  $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$  the  $p$ -adic completion of  $L$  and by  $b_{p^k}^{\wedge}(L_p)$  the number of Lie sublattices of  $L_p$  of index  $p^k$  that are isomorphic to  $L_p$ . Then the local zeta function  $\zeta_{L_p}^{\wedge}(s) = \sum_{k=0}^{\infty} b_{p^k}^{\wedge}(L_p) p^{-ks}$  coincides with the local zeta function  $\zeta_{\Delta,p}^{\wedge}(s)$  of the group  $\Delta$ .

We fix a field  $K$  and put  $L_K = K \otimes_{\mathbb{Z}} L$ . Our task is to determine the automorphism group  $\text{Aut}(L_K)$  as a subgroup of  $\text{GL}(L_K)$ , with reference to the basis (2.1).

**Lemma 2.3.** *The  $K$ -space  $A = \langle y_{\mathbf{f}} \mid \mathbf{f} \in \mathbf{F} \rangle + Z(L_K)$  constitutes the unique abelian Lie-ideal of dimension  $\binom{m+n-1}{n-1} + n$  in  $L_K$ . Consequently,  $A$  is  $\text{Aut}(L_K)$ -invariant.*



*Proof.* Clearly,  $A$  is an abelian ideal of dimension  $\binom{m+n-1}{n-1} + n$  in  $L_K$ . Suppose that  $B$  is a maximal abelian Lie-ideal of  $L_K$  and  $B \not\subseteq A$ . It suffices to show that  $\dim_K(B) < \dim_K(A)$ . Clearly,  $Z(L_K) \subseteq B$ . We work in  $\overline{L_K} = L_K/Z(L_K) = \overline{V} \oplus \overline{W}$ , where  $\overline{V} = \langle \overline{x_e} \mid e \in \mathbf{E} \rangle$  and  $\overline{W} = \langle \overline{y_f} \mid f \in \mathbf{F} \rangle$ . As  $B \not\subseteq A$ , we have  $\overline{V_1} \neq 0$ , where  $\overline{V_1} \leq \overline{V}$  is such that  $\overline{V_1} \oplus \overline{W} = \overline{B} + \overline{W}$ .

We construct a  $K$ -basis  $\mathcal{B} = (\overline{v_1(1)}, \dots, \overline{v_1(r)})$  of  $\overline{V_1}$  as follows. Fix  $i \in \{1, \dots, n\}$ . We define a lexicographical order on the set  $\mathbf{E}$  by declaring, for  $e, e' \in \mathbf{E}$ ,

$$e \succ_i e' \quad \text{if } e_i = e'_i, e_{i+1} = e'_{i+1}, \dots, e_j = e'_j \text{ and } e_{j+1} > e'_{j+1} \\ \text{for some } j \in \{i-1, i, \dots, n, 1, 2, \dots, i-2\},$$

where – here and in the following – indices for the coordinates of  $e$  and  $e'$  are read in a circular way modulo  $n$ . Accordingly, we write

$$e \succeq_i e' \quad \text{if } e = e' \text{ or } e \succ_i e'.$$

Of course, one can define analogously an order on  $\mathbf{F}$  which we shall simply refer to by the same symbols  $\succ_i$  and  $\succeq_i$ . By  $\text{lt}_i(\overline{v})$  we denote the leading term of  $\overline{v} \in \overline{V}$  with respect to the ordered basis  $(\overline{x_e} \mid e \in \mathbf{E}; \succeq_i)$ .

We now choose the basis  $\mathcal{B} = (\overline{v_1(1)}, \dots, \overline{v_1(r)})$  of  $\overline{V_1}$  in such a way that

$$\text{lt}_i(\overline{v_1(j)}) = \overline{x_{e(j)}} \quad \text{for } 1 \leq j \leq r,$$

where  $e(1) \succ_i e(2) \succ_i \dots \succ_i e(r)$  and the coefficient matrix of the vector system  $(\overline{v_1(1)}, \dots, \overline{v_1(r)})$  with respect to the basis  $(\overline{x_e} \mid e \in \mathbf{E}; \succeq_i)$  of  $\overline{V}$  has reduced echelon shape

$$\begin{pmatrix} 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots \\ 0 & \dots & & & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots \\ 0 & \dots & & & \dots & & \dots & 0 & 1 & * & \dots \\ \dots & & & \dots & & & \dots & & \dots & & \dots \end{pmatrix}.$$

Next we consider  $\overline{V_1}^\perp \leq \overline{W}$ , where

$$\overline{V_1}^\perp = \{\overline{w} \in \overline{W} \mid \forall j \in \{1, \dots, r\} : [\overline{v_1(j)}, \overline{w}] = 0\}$$

is defined in terms of the induced ‘Lie bracket map’  $L_K/Z(L_K) \times L_K/Z(L_K) \rightarrow Z(L_K)$ .

Below we show that

$$(2.2) \quad \dim_K(\overline{V_1}^\perp) < \dim_K(\overline{W}) - \dim_K(\overline{V_1}).$$

We observe that  $\dim_K(\overline{B}) = \dim_K(\overline{V_1}) + \dim_K(\overline{B} \cap \overline{W})$  and  $\overline{B} \cap \overline{W} \subseteq \overline{V_1}^\perp$ , the latter because  $B$  is an abelian Lie-ideal. This implies

$$\dim_K(\overline{B}) \leq \dim_K(\overline{V_1}) + \dim_K(\overline{V_1}^\perp) < \dim_K(\overline{W}),$$

thus  $\dim_K(B) < \dim_K(A)$ , as desired.

It remains to justify (2.2). For this it suffices to produce  $\overline{w_1}, \dots, \overline{w_{r+1}} \in \overline{W}$  such that

$$\dim_K(\langle \overline{w_1}, \dots, \overline{w_{r+1}} \rangle) = r + 1 = \dim_K(\overline{V_1}) + 1 \quad \text{and} \quad \langle \overline{w_1}, \dots, \overline{w_{r+1}} \rangle \cap \overline{V_1}^\perp = 0.$$

Take  $\overline{w_j} = \overline{y_{f(j)}}$  for  $1 \leq j \leq r$ , where

$$\mathbf{f}(j) = \mathbf{e}(j) + (\underbrace{0, \dots, 0}_{i-1 \text{ entries}}, 1, \underbrace{0, \dots, 0}_{n-i \text{ entries}})$$

and  $\overline{w_{r+1}} = \overline{y_{f(r+1)}}$ , where

$$\mathbf{f}(r+1) = \tilde{\mathbf{e}} + (\underbrace{0, \dots, 0}_{i-2 \text{ entries}}, 1, \underbrace{0, \dots, 0}_{n-i+1 \text{ entries}})$$

and  $\tilde{\mathbf{e}}$  is the  $\succeq_i$ -smallest index such that at least one of  $\overline{v_1(1)}, \dots, \overline{v_1(r)}$  has a non-zero coefficient for  $\overline{x_{\tilde{\mathbf{e}}}}$  with respect to the basis  $(\overline{x_{\mathbf{e}}} \mid \mathbf{e} \in \mathbf{E}; \succeq_i)$  of  $\overline{V}$ .

To see that  $\overline{w_1}, \dots, \overline{w_{r+1}}$  are linearly independent, it suffices to show that the indices  $\mathbf{f}(1), \dots, \mathbf{f}(r+1)$  are pairwise distinct. Clearly, this is true for  $\mathbf{f}(1), \dots, \mathbf{f}(r)$ , because  $\mathbf{e}(1), \dots, \mathbf{e}(r)$  are pairwise distinct by construction. It remains to show that  $\mathbf{f}(j) \neq \mathbf{f}(r+1)$  for  $1 \leq j \leq r$ . This follows from  $\mathbf{e}(j) \succeq_i \tilde{\mathbf{e}}$ , implying  $\mathbf{f}(j) \succ_i \mathbf{f}(r+1)$ .

To show that

$$\langle \overline{w_1}, \dots, \overline{w_{r+1}} \rangle \cap \overline{V_1}^\perp = 0,$$

we consider  $\overline{w} = \sum_{j=1}^{r+1} a_j \overline{w_j} \neq 0$ . Put  $k = \min\{j \mid a_j \neq 0\}$ .

*Case 1:*  $1 \leq k \leq r$ . Then there are coefficients  $b_j$ , for  $j \neq i$ , such that

$$[\overline{v_1(k)}, \overline{w}] = [\overline{x_{\mathbf{e}(k)}} + \dots, \overline{w}] = \underbrace{a_k}_{\neq 0} \underbrace{[\overline{x_{\mathbf{e}(k)}}, \overline{y_{\mathbf{f}(k)}}]}_{=z_i} + \sum_{j \neq i} b_j z_j \neq 0,$$

because  $\overline{v_1(k)}$  does not involve  $\overline{x_{\mathbf{e}(j)}}$  for  $j \in \{1, \dots, r\} \setminus \{k\}$  or  $\overline{x_{\tilde{\mathbf{e}}}}$  for  $\tilde{\mathbf{e}} \succ_i \mathbf{e}$ .

*Case 2:*  $k = r+1$ . In this case  $\overline{w} = a_{r+1} \overline{y_{\mathbf{f}(r+1)}}$  and  $\overline{x_{\tilde{\mathbf{e}}}}$  occurs in  $\overline{v_1(j)}$ , say, with coefficient  $c \neq 0$ . Then there are coefficients  $b_l$ ,  $l \neq i-1$ , such that

$$[\overline{v_1(j)}, \overline{w}] = \underbrace{ca_{r+1}}_{\neq 0} \underbrace{[\overline{x_{\tilde{\mathbf{e}}}}, \overline{y_{\mathbf{f}(r+1)}}]}_{=z_{i-1}} + \sum_{l \neq i-1} b_l z_l \neq 0. \quad \square$$

**Lemma 2.4.** *Let  $\varphi \in \text{Aut}(L_K)$  with  $\varphi|_{Z(L_K)} = \text{id}$ . Then there exist  $\lambda \in K^\times$  and matrices  $C \in \text{Mat}_{|\mathbf{E}|, |\mathbf{F}|}(K)$ ,  $D_1 \in \text{Mat}_{|\mathbf{E}|, n}(K)$ ,  $D_2 \in \text{Mat}_{|\mathbf{F}|, n}(K)$  such that, with respect to the basis (2.1), the automorphism  $\varphi$  is represented by the block-matrix*

$$\begin{pmatrix} \lambda \text{Id}_{|\mathbf{E}|} & C & D_1 \\ & \lambda^{-1} \text{Id}_{|\mathbf{F}|} & D_2 \\ & & \text{Id}_n \end{pmatrix},$$

where empty positions represent zeros.

*Proof.* By Lemma 2.3, there exist matrices  $M_1 \in \text{Mat}_{|\mathbf{E}|, |\mathbf{E}|}(K)$ ,  $M_2 \in \text{Mat}_{|\mathbf{F}|, |\mathbf{F}|}(K)$ ,  $C \in \text{Mat}_{|\mathbf{E}|, |\mathbf{F}|}(K)$ ,  $D_1 \in \text{Mat}_{|\mathbf{E}|, n}(K)$ ,  $D_2 \in \text{Mat}_{|\mathbf{F}|, n}(K)$  such that, with respect to the basis (2.1), the automorphism  $\varphi$  is represented by the block-matrix

$$\begin{pmatrix} M_1 & C & D_1 \\ & M_2 & D_2 \\ & & \text{Id}_n \end{pmatrix}.$$

We show that there exists  $\lambda \in K^\times$  such that  $M_1 = \lambda \text{Id}_{|\mathbf{E}|}$  and  $M_2 = \lambda^{-1} \text{Id}_{|\mathbf{F}|}$ .

Working modulo the centre  $Z(L_K)$ , we obtain vector spaces  $\overline{V} = \langle \overline{x_e} \mid e \in \mathbf{E} \rangle$  and  $\overline{W} = \langle \overline{y_f} \mid f \in \mathbf{F} \rangle$ , equipped with bilinear forms

$$B_1, \dots, B_n: \overline{V} \times \overline{W} \rightarrow K$$

such that the induced Lie bracket map satisfies  $[\overline{v}, \overline{w}] = \sum_{j=1}^n B_j(\overline{v}, \overline{w}) z_j$  for  $\overline{v} \in \overline{V}$  and  $\overline{w} \in \overline{W}$ .

Since  $[\overline{W}, \overline{W}] = 0$ , we may in the following argument ‘ignore’  $C$  and assume that  $\varphi$  maps each of  $\overline{V}, \overline{W}$  to itself. Since  $\varphi$  restricts to the identity on  $Z(L_K)$ , we deduce that  $\varphi$  preserves the bilinear forms  $B_1, \dots, B_n$ . Below we show that  $\varphi|_{\overline{V}} = \lambda \text{id}$  for some  $\lambda \in K^\times$ . This implies  $\varphi|_{\overline{W}} = \lambda^{-1} \text{id}$ , since the intersection of the (right) radicals

$$\overline{R_i} = \text{Rad}_{\overline{W}}(B_i) = \langle y_f \mid f = (f_1, \dots, f_n) \in \mathbf{F} \text{ with } f_i = 0 \rangle$$

for  $i \in \{1, \dots, n\}$  is trivial.

For  $m = 1$  we have  $|\mathbf{E}| = 1$  and there is nothing further to show. Now suppose that  $m \geq 2$ , and fix  $i \in \{1, \dots, n\}$ . The radical  $\overline{R_i}$  is a  $\varphi$ -invariant subspace of  $\overline{W}$ . Consequently, also

$$\begin{aligned} \overline{U_i} &= \bigcap_{j=1}^n \overline{R_i}^{\perp, j} = \{ \overline{v} \in \overline{V} \mid \forall j \in \{1, \dots, n\} : B_j(\overline{v}, \overline{R_i}) = 0 \} \\ &= \langle \overline{x_e} \mid e = (e_1, \dots, e_n) \in \mathbf{E} \text{ with } e_i \geq 1 \rangle \end{aligned}$$

is  $\varphi$ -invariant.

Put  $\overline{V_1} = \overline{U_i}$  and  $\overline{W_1} = \overline{W}/\overline{R_i}$ , equipped with the induced bilinear forms

$$\widetilde{B}_1, \dots, \widetilde{B}_n: \overline{V_1} \times \overline{W_1} \rightarrow K$$

and an induced automorphism  $\widetilde{\varphi}$ . By induction on  $m$ , we conclude that there exists  $\lambda_i \in K^\times$  such that

$$\varphi|_{\overline{U_i}} = \widetilde{\varphi}|_{\overline{V_1}} = \lambda_i \text{id}$$

and, though we will not use this,  $\widetilde{\varphi}|_{\overline{W_1}} = \lambda_i^{-1} \text{id}$ , i.e.,  $\widetilde{\varphi}|_{\overline{W}} \equiv \lambda_i^{-1} \text{id}$  modulo  $\overline{R_i}$ .

Repeating this argument for different  $i \in \{1, \dots, n\}$  and comparing on the intersection of the  $\overline{U_i}$ , we deduce that  $\varphi|_{\overline{V}} = \lambda \text{id}$  for a common  $\lambda \in K^\times$ .  $\square$

**Lemma 2.5.** *Let  $A = \langle y_f \mid f \in \mathbf{F} \rangle + Z(L_K) \trianglelefteq L_K$ , as in Lemma 2.3. Suppose that  $\varphi \in \text{Aut}(L_K)$  induces the identity map on  $L_K/A$ ,  $A/Z(L_K)$  and  $Z(L_K)$ . Then there exist matrices  $C \in \text{Mat}_{|\mathbf{E}|, |\mathbf{F}|}(K)$ ,  $D_1 \in \text{Mat}_{|\mathbf{E}|, n}(K)$ ,  $D_2 \in \text{Mat}_{|\mathbf{F}|, n}(K)$  such that, with respect to the basis (2.1), the automorphism  $\varphi$  is represented by the block-matrix*

$$(2.3) \quad \begin{pmatrix} \text{Id}_{|\mathbf{E}|} & C & D_1 \\ & \text{Id}_{|\mathbf{F}|} & D_2 \\ & & \text{Id}_n \end{pmatrix},$$

where empty positions represent zeros. The rows of the matrix  $C$  are naturally indexed by  $e \in \mathbf{E}$ ; its columns are naturally indexed by  $f \in \mathbf{F}$ . Writing  $C = (c_{e,f})_{(e,f) \in \mathbf{E} \times \mathbf{F}}$ , there are parameters  $b_g$  indexed by the elements of

$$\mathbf{G} = \{ \mathbf{g} \mid \mathbf{g} = (g_1, \dots, g_n) \in \mathbb{N}_0^n \text{ with } g_1 + \dots + g_n = 2m - 1 \}$$

such that  $c_{e,f} = b_{e+f}$  for all  $e \in \mathbf{E}$ ,  $f \in \mathbf{F}$ .

Conversely, given matrices  $C = (c_{\mathbf{e},\mathbf{f}})_{(\mathbf{e},\mathbf{f}) \in \mathbf{E} \times \mathbf{F}} \in \text{Mat}_{|\mathbf{E}|, |\mathbf{F}|}(K)$ ,  $D_1 \in \text{Mat}_{|\mathbf{E}|, n}(K)$ ,  $D_2 \in \text{Mat}_{|\mathbf{F}|, n}(K)$  such that  $c_{\mathbf{e},\mathbf{f}} = c_{\mathbf{e}',\mathbf{f}'}$  for  $\mathbf{e}, \mathbf{e}' \in \mathbf{E}$  and  $\mathbf{f}, \mathbf{f}' \in \mathbf{F}$  with  $\mathbf{e} + \mathbf{f} = \mathbf{e}' + \mathbf{f}'$  there is an automorphism of  $L_K$  that is represented by the block matrix (2.3).

*Proof.* We indicate how to prove the first part; the second part is a routine verification. The assumptions on  $\varphi$  show directly that  $\varphi$  is represented by a block matrix of the form (2.3). In particular, this means that, for each  $\mathbf{e} \in \mathbf{E}$ ,

$$x_{\mathbf{e}}\varphi \equiv x_{\mathbf{e}} + \sum_{\mathbf{f} \in \mathbf{F}} c_{\mathbf{e},\mathbf{f}} y_{\mathbf{f}} \pmod{Z(L_K)}.$$

Since  $\varphi$  is an automorphism, we conclude that, for  $\mathbf{e}, \mathbf{e}' \in \mathbf{E}$ ,

$$\left[ x_{\mathbf{e}} + \sum_{\mathbf{f} \in \mathbf{F}} c_{\mathbf{e},\mathbf{f}} y_{\mathbf{f}}, x_{\mathbf{e}'} + \sum_{\mathbf{f}' \in \mathbf{F}} c_{\mathbf{e}',\mathbf{f}'} y_{\mathbf{f}'} \right] = [x_{\mathbf{e}}\varphi, x_{\mathbf{e}'}\varphi] = [x_{\mathbf{e}}, x_{\mathbf{e}'}]\varphi = 0,$$

and consequently  $c_{\mathbf{e},\mathbf{f}} = c_{\mathbf{e}',\mathbf{f}'}$ , for  $\mathbf{f}, \mathbf{f}' \in \mathbf{F}$ , whenever

$$(2.4) \quad \mathbf{f} - \mathbf{e}' = \mathbf{f}' - \mathbf{e} = (\underbrace{0, \dots, 0}_{i-1 \text{ entries}}, 1, \underbrace{0, \dots, 0}_{n-i \text{ entries}}) \quad \text{for some } i \in \{1, \dots, n\}.$$

We claim that this implies – and is thus equivalent to – the condition

$$c_{\mathbf{e},\mathbf{f}} = c_{\mathbf{e}',\mathbf{f}'} \quad \text{whenever } \mathbf{e} + \mathbf{f} = \mathbf{e}' + \mathbf{f}'.$$

Indeed, given any  $\mathbf{e} = (e_1, \dots, e_n), \mathbf{e}' = (e'_1, \dots, e'_n) \in \mathbf{E}$  and  $\mathbf{f}, \mathbf{f}' \in \mathbf{F}$  with  $\mathbf{e} + \mathbf{f} = \mathbf{e}' + \mathbf{f}'$  one can find a chain

$$(\mathbf{e}, \mathbf{f}) = (\mathbf{e}_1, \mathbf{f}_1), (\mathbf{e}_2, \mathbf{f}_2), \dots, (\mathbf{e}_{\delta}, \mathbf{f}_{\delta}), (\mathbf{e}_{\delta+1}, \mathbf{f}_{\delta+1}) = (\mathbf{e}', \mathbf{f}')$$

of length  $\delta = \sum_{i=1}^n |e_i - e'_i|$  such that the condition (2.4) is satisfied for any two consecutive terms  $(\mathbf{e}_t, \mathbf{f}_t), (\mathbf{e}_{t+1}, \mathbf{f}_{t+1})$  of the chain so that

$$c_{\mathbf{e},\mathbf{f}} = c_{\mathbf{e}_1,\mathbf{f}_1} = c_{\mathbf{e}_2,\mathbf{f}_2} = \dots = c_{\mathbf{e}_{\delta},\mathbf{f}_{\delta}} = c_{\mathbf{e}_{\delta+1},\mathbf{f}_{\delta+1}} = c_{\mathbf{e}',\mathbf{f}'}.$$

To produce such a chain, we observe that  $\delta \equiv_2 0$  so that we can move inductively in steps of two. Suppose we have reached  $(\mathbf{e}_t, \mathbf{f}_t)$ , not yet equal to  $(\mathbf{e}', \mathbf{f}')$ . Writing  $\mathbf{e}_t = (e_{t,1}, \dots, e_{t,n})$ , we locate  $j, k \in \{1, \dots, n\}$  such that  $e_{t,j} < e'_j$  and  $e_{t,k} > e'_k$ . Using (2.4) at  $j$  and  $k$  in place of  $i$ , we reach via some intermediate  $(\mathbf{e}_{t+1}, \mathbf{f}_{t+1})$  the next term  $(\mathbf{e}_{t+2}, \mathbf{f}_{t+2})$ , where  $\mathbf{e}_{t+2} = (e_{t+2,1}, \dots, e_{t+2,n})$  satisfies:  $e_{t+2,j} = e_{t,j} + 1$ ,  $e_{t+2,k} = e_{t,k} - 1$  and  $e_{t+2,i} = e_{t,i}$  for all remaining indices  $i$ . As  $\mathbf{e}_t + \mathbf{f}_t$  remains constant throughout, at the end of the final step  $\mathbf{e}_{\delta+1} = \mathbf{e}'$  automatically implies  $\mathbf{f}_{\delta+1} = \mathbf{f}'$ .  $\square$

**Lemma 2.6.** *Suppose that the characteristic of  $K$  is 0. Then every  $\psi \in \text{GL}(Z(L_K))$  extends to an automorphism  $\varphi \in \text{Aut}(L_K)$  so that  $\varphi|_{Z(L_K)} = \psi$ . Moreover,  $\varphi$  can be chosen so that the spaces  $\langle x_{\mathbf{e}} \mid \mathbf{e} \in \mathbf{E} \rangle$  and  $\langle y_{\mathbf{f}} \mid \mathbf{f} \in \mathbf{F} \rangle$  are  $\varphi$ -invariant and the action of  $\varphi$  on them corresponds to induced actions of  $\psi$  on symmetric powers of the dual space of  $Z(L_K)$  and on symmetric powers of  $Z(L_K)$ , respectively.*

*Proof.* Regard  $Z(L_K) = \langle z_1, \dots, z_n \rangle$  as the dual space  $V^{\vee}$  of an  $n$ -dimensional  $K$ -vector space  $V = \langle \xi_1, \dots, \xi_n \rangle$  such that, for  $i, j \in \{1, \dots, n\}$ ,

$$(\xi_i)z_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The  $(m-1)$ th symmetric power  $S^{m-1}V$  of  $V$  can be constructed as  $V^{\otimes m-1}/\text{Sym}(m-1)$ , where  $(\nu_1 \otimes \dots \otimes \nu_{m-1})^\sigma = \nu_{1\sigma^{-1}} \otimes \dots \otimes \nu_{(m-1)\sigma^{-1}}$  for  $\sigma \in \text{Sym}(m-1)$ . Writing  $\nu_1 \dots \nu_{m-1}$  for the element of  $S^{m-1}V$  represented by  $\nu_1 \otimes \dots \otimes \nu_{m-1}$ , one easily sees that a basis of  $S^{m-1}V$  is given by the vectors  $\xi_1^{e_1} \dots \xi_n^{e_n}$ ,  $\mathbf{e} = (e_1, \dots, e_n) \in \mathbf{E}$ . In a similar way, we form the  $m$ th symmetric power  $S^m(V^\vee)$  with basis  $z_1^{f_1} \dots z_n^{f_n}$ ,  $\mathbf{f} \in \mathbf{F}$ .

One can interpret  $S^m(V^\vee)$  as  $(S^m V)^\vee$ , and this suggests a change of basis. Indeed, the natural map

$$V^{\otimes m} \times (V^\vee)^{\otimes m} \rightarrow K, \quad \text{induced by } (\nu_1 \otimes \dots \otimes \nu_m)(a_1 \otimes \dots \otimes a_m) = \prod_{i=1}^m (\nu_i) a_i$$

gives rise to an isomorphism  $(V^\vee)^{\otimes m} \cong (V^{\otimes m})^\vee$ . Now  $\xi_1^{f'_1} \dots \xi_n^{f'_n} \in S^m V$  is represented, for instance, by

$$\underbrace{\xi_1 \otimes \dots \otimes \xi_1}_{f'_1 \text{ factors}} \otimes \dots \otimes \underbrace{\xi_n \otimes \dots \otimes \xi_n}_{f'_n \text{ factors}} \in V^{\otimes m}.$$

Moreover, as  $K$  has characteristic 0, there is an embedding

$$S^m(V^\vee) \hookrightarrow (V^\vee)^{\otimes m}, \quad z_1^{f_1} \dots z_n^{f_n} \mapsto \sum_{\sigma \in \text{Sym}(m)} \underbrace{(z_1 \otimes \dots \otimes z_1)_{f_1 \text{ factors}}} \otimes \dots \otimes \underbrace{(z_n \otimes \dots \otimes z_n)_{f_n \text{ factors}}}^\sigma;$$

the image of this embedding consists of the elements of  $(V^\vee)^{\otimes m}$  that are fixed under the action of  $\text{Sym}(m)$ . Hence we can evaluate (the image of)  $z_1^{f_1} \dots z_n^{f_n} \in S^m(V^\vee)$  (under this embedding) at  $\xi_1^{f'_1} \dots \xi_n^{f'_n} \in S^m V$  to obtain

$$(\xi_1^{f'_1} \dots \xi_n^{f'_n})(z_1^{f_1} \dots z_n^{f_n}) = \begin{cases} \prod_{i=1}^n (f_i!) & \text{if } (f'_1, \dots, f'_n) = (f_1, \dots, f_n), \\ 0 & \text{otherwise} \end{cases}$$

Therefore the basis  $\xi_1^{f'_1} \dots \xi_n^{f'_n}$ , indexed by  $\mathbf{f}' = (f'_1, \dots, f'_n) \in \mathbf{F}$ , of  $S^m V$  gives rise to the dual basis  $(\prod_{i=1}^n (f_i!))^{-1} z_1^{f_1} \dots z_n^{f_n}$ , indexed by  $\mathbf{f} = (f_1, \dots, f_n) \in \mathbf{F}$ , of  $S^m(V^\vee)$ , and this is the basis that we prefer to work with in the present context.

We define a multilinear map

$$\Phi: S^{m-1}V \times \underbrace{V}_{=S^1 V} \times S^m(V^\vee) \rightarrow K$$

by setting

$$\begin{aligned} \Phi \left( \xi_1^{e_1} \dots \xi_n^{e_n}, \xi_k, \left( \prod_{i=1}^n (f_i!) \right)^{-1} z_1^{f_1} \dots z_n^{f_n} \right) &= (\xi_1^{\tilde{e}_1} \dots \xi_n^{\tilde{e}_n}) \left( \left( \prod_{i=1}^n (f_i!) \right)^{-1} z_1^{f_1} \dots z_n^{f_n} \right) \\ &= \begin{cases} 1 & \text{if } (\tilde{e}_1, \dots, \tilde{e}_n) = (f_1, \dots, f_n), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where  $\tilde{e}_k = e_k + 1$  and  $\tilde{e}_j = e_j$  for  $j \neq k$ , and extending linearly in each of the three arguments. This map  $\Phi$  connects to the original Lie algebra  $L_K$  as follows. For

$\mathbf{e} = (e_1, \dots, e_n) \in \mathbf{E}$  and  $\mathbf{f} = (f_1, \dots, f_n) \in \mathbf{F}$ ,

$$\begin{aligned} [x_{\mathbf{e}}, y_{\mathbf{f}}] &= \sum_{k=1}^n \Phi \left( \xi_1^{e_1} \cdots \xi_n^{e_n}, \xi_k, \left( \prod_{i=1}^n (f_i!) \right)^{-1} z_1^{f_1} \cdots z_n^{f_n} \right) \cdot z_k \\ &= \Phi \left( \xi_1^{e_1} \cdots \xi_n^{e_n}, \cdot, \left( \prod_{i=1}^n (f_i!) \right)^{-1} z_1^{f_1} \cdots z_n^{f_n} \right) \in V^\vee = Z(L_K). \end{aligned}$$

The fact that  $\Phi$  is multilinear means that we can monitor changes to a set of structure constants for  $L_K$  due to changes to any of the three parts of the basis

$$x_{\mathbf{e}}, \mathbf{e} \in \mathbf{E}, \quad y_{\mathbf{f}}, \mathbf{f} \in \mathbf{F}, \quad z_j, j \in \{1, \dots, n\}.$$

In particular, given any  $\psi \in \mathrm{GL}(V^\vee)$ , there are the natural induced actions of  $\psi$  on

- $V^\vee$  via the natural representation,
- $V$  via the contragredient representation (given in matrices by inverse transpose) which we denote by  $\psi^*$ ,
- $S^m(V^\vee)$ , sending  $(\prod_{j=1}^n (f_j!))^{-1} z_1^{f_1} \cdots z_n^{f_n}$  to  $(\prod_{j=1}^n (f_j!))^{-1} (z_1 \psi)^{f_1} \cdots (z_n \psi)^{f_n}$ ,
- $S^{m-1}V$ , sending  $\xi_1^{e_1} \cdots \xi_n^{e_n}$  to  $(\xi_1 \psi^*)^{e_1} \cdots (\xi_n \psi^*)^{e_n}$ .

In this set-up we clearly obtain

$$\begin{aligned} &\Phi \left( \xi_1^{e_1} \cdots \xi_n^{e_n}, \xi_k, \left( \prod_{i=1}^n (f_i!) \right)^{-1} z_1^{f_1} \cdots z_n^{f_n} \right) \\ &= \Phi \left( (\xi_1 \psi^*)^{e_1} \cdots (\xi_n \psi^*)^{e_n}, (\xi_k \psi^*), \left( \prod_{i=1}^n (f_i!) \right)^{-1} (z_1 \psi)^{f_1} \cdots (z_n \psi)^{f_n} \right), \end{aligned}$$

because the bases  $\xi_1 \psi^*, \dots, \xi_n \psi^*$  of  $V$  and  $z_1 \psi, \dots, z_n \psi$  of  $V^\vee$  are dual to one another by construction. Thus the given  $\psi \in \mathrm{GL}(V^\vee) = \mathrm{GL}(Z(L_K))$  extends to an automorphism  $\varphi \in \mathrm{Aut}(L_K)$  so that  $\varphi|_{Z(L_K)} = \psi$ ; moreover,  $\varphi$  acts on the spaces  $\langle x_{\mathbf{e}} \mid \mathbf{e} \in \mathbf{E} \rangle$  and  $\langle y_{\mathbf{f}} \mid \mathbf{f} \in \mathbf{F} \rangle$  as  $\psi$  does on the corresponding symmetric powers.  $\square$

From Lemmata 2.4, 2.5 and 2.6, including the proof of Lemma 2.6, we deduce the following result.

**Theorem 2.7.** *Let  $m, n \in \mathbb{N}$  with  $n \geq 2$ , and let  $L = L_{m,n}$  be the class-2 nilpotent  $\mathbb{Z}$ -Lie lattice introduced in Definition 2.1, so that  $\mathrm{rank}_{\mathbb{Z}} L = d = r_1 + r_2 + r_3$ , where*

$$r_1 = |\mathbf{E}| = \binom{m+n-2}{n-1}, \quad r_2 = |\mathbf{F}| = \binom{m+n-1}{n-1}, \quad r_3 = n.$$

*Then the algebraic automorphism group  $\mathrm{Aut}(L)$ , regarded as an affine  $\mathbb{Z}$ -group scheme  $\mathbf{G} \subseteq \mathrm{GL}_d$  with respect to the basis (2.1), has the following structure:  $\mathbf{G}$  decomposes over  $\mathbb{Z}$  as a semidirect product  $\mathbf{G} = \mathbf{N} \rtimes \mathbf{H}$  of its unipotent radical  $\mathbf{N}$  and a reductive complement  $\mathbf{H}$ , both defined over  $\mathbb{Z}$ , such that*

- $\mathbf{N}$  consists of elements of the form

$$\begin{pmatrix} \mathrm{Id}_{|\mathbf{E}|} & C & D_1 \\ & \mathrm{Id}_{|\mathbf{F}|} & D_2 \\ & & \mathrm{Id}_n \end{pmatrix},$$

*where  $C \in \mathrm{Mat}_{|\mathbf{E}|, |\mathbf{F}|}$ ,  $D_1 \in \mathrm{Mat}_{|\mathbf{E}|, n}$ ,  $D_2 \in \mathrm{Mat}_{|\mathbf{F}|, n}$  subject to the following conditions. The rows of  $C$  are naturally indexed by  $\mathbf{e} \in \mathbf{E}$ , its columns are naturally indexed by*

$\mathbf{f} \in \mathbf{F}$ . Writing  $C = (c_{\mathbf{e},\mathbf{f}})_{(\mathbf{e},\mathbf{f}) \in \mathbf{E} \times \mathbf{F}}$ , there are parameters  $b_{\mathbf{g}}$  indexed by the elements of

$$\mathbf{G} = \{\mathbf{g} \mid \mathbf{g} = (g_1, \dots, g_n) \in \mathbb{N}_0^n \text{ with } g_1 + \dots + g_n = 2m - 1\}$$

such that  $c_{\mathbf{e},\mathbf{f}} = b_{\mathbf{e}+\mathbf{f}}$  for all  $(\mathbf{e}, \mathbf{f}) \in \mathbf{E} \times \mathbf{F}$ .

- $H \cong_{\mathbb{Z}} \mathrm{GL}_n \times \mathrm{GL}_1$  consists of elements of the form

$$\begin{pmatrix} \lambda \varrho_1(A) & & \\ & \lambda^{-1} \varrho_2(A) & \\ & & A \end{pmatrix},$$

where  $A \in \mathrm{GL}_n$ ,  $\lambda \in \mathbb{G}_m$  and  $\varrho_1, \varrho_2$  are suitable symmetric power representations of  $\mathrm{GL}_n$  in dimensions  $r_1$  and  $r_2$ .

### 3. EVALUATING $p$ -ADIC INTEGRALS VIA THE BRUHAT DECOMPOSITION

In [6, Sections 2], du Sautoy and Lubotzky provide a general treatment of  $\mathfrak{p}$ -adic integrals of the form (1.2), subject to several simplifying assumptions. We now give an outline of the procedure and indicate how it applies to the specific affine group scheme supplied by Theorem 2.7. It will turn out that for our application all necessary assumptions are automatically satisfied.

Let  $G \subseteq \mathrm{GL}_d$  be an affine group scheme over the ring of integers  $\mathcal{O}$  of a number field  $\mathcal{K}$ . Decompose the connected component of the identity as a semidirect product  $G^\circ = N \rtimes H$  of the unipotent radical  $N$  and a reductive complement  $H$ . For a finite prime  $\mathfrak{p}$  let  $\mathcal{K}_{\mathfrak{p}}$  denote the completion at  $\mathfrak{p}$ , and let  $\mathcal{O}_{\mathfrak{p}}$  denote the valuation ring of  $\mathcal{K}_{\mathfrak{p}}$ . Fix a uniformising parameter  $\pi$  for  $\mathcal{O}_{\mathfrak{p}}$ , and let  $q$  denote the size of the residue field  $\mathcal{O}_{\mathfrak{p}}/\pi\mathcal{O}_{\mathfrak{p}}$ . Setting  $G = G(\mathcal{K}_{\mathfrak{p}})$  and  $G^+ = G \cap \mathrm{Mat}_d(\mathcal{O}_{\mathfrak{p}})$ , we are interested in the zeta function

$$\mathcal{Z}_{G,\mathfrak{p}}(s) = \int_{G^+} |\det(g)|_{\mathfrak{p}}^s d\mu_G(g).$$

The first simplifying assumption is that  $G = G(\mathcal{O}_{\mathfrak{p}}) G^\circ(\mathcal{K}_{\mathfrak{p}})$ , which allows one to work with the connected group  $G^\circ$  rather than  $G$ . In our application, the group  $G$  is already connected and we continue the discussion under the assumption  $G = G^\circ$ . We write

$$N = N(\mathcal{K}_{\mathfrak{p}}) \quad \text{and} \quad H = H(\mathcal{K}_{\mathfrak{p}}).$$

Assume further that  $G \subseteq \mathrm{GL}_d(\mathcal{K}_{\mathfrak{p}})$  is in block form, where  $H$  is block diagonal and  $N$  is block upper unitriangular in the following sense. There is a partition  $d = r_1 + \dots + r_c$  such that, setting  $s_i = r_1 + \dots + r_{i-1}$  for  $i \in \{1, \dots, c+1\}$ ,

- the vector space  $V = \mathcal{K}_{\mathfrak{p}}^d$  on which  $G$  acts from the right decomposes into a direct sum of  $H$ -stable subspaces  $U_i = \{(0, \dots, 0)\} \times \mathcal{K}_{\mathfrak{p}}^{r_i} \times \{(0, \dots, 0)\}$ , where the vectors  $(0, \dots, 0)$  have  $s_i$ , respectively  $d - s_{i+1}$ , entries;
- setting  $V_i = U_i \oplus \dots \oplus U_c$ , each  $V_i$  is  $N$ -stable and  $N$  acts trivially on  $V_i/V_{i+1}$ .

In our application, Theorem 2.7 provides such a block decomposition with  $c = 3$ . For each  $i \in \{2, \dots, c+1\}$  let  $N_{i-1} = N \cap \ker(\psi'_i)$ , where  $\psi'_i: G \rightarrow \mathrm{Aut}(V/V_i)$  denotes the natural action. Let  $\psi_i: G/N_{i-1} \rightarrow \mathrm{Aut}(V/V_i)$  denote the induced map, and define

$$(G/N_{i-1})^+ = \psi_i^{-1}(\psi_i(G/N_{i-1}) \cap \mathrm{Mat}_{s_i}(\mathcal{O}_{\mathfrak{p}})).$$



Setting  $H^+ = H \cap \text{Mat}_d(\mathcal{O}_{\mathfrak{p}})$ , we can now state (a slightly simplified version of) the ‘lifting condition’, which cannot in general be satisfied by moving to an equivalent representation; see [2, p. 6].

**‘Lifting Condition’** (cf. [6, Assumption 2.3]). For each  $i \in \{3, \dots, c\}$  and every  $g_0 N_{i-1} \in (NH^+/N_{i-1})^+$  there exists  $g \in G^+$  such that  $g_0 N_{i-1} = g N_{i-1}$ .

In our application,  $c = 3$  is so small that the ‘lifting condition’ holds for trivial reasons: according to the description given in Theorem 2.7, the relevant blocks  $D_1$  and  $D_2$  of any element of  $N$  can just be replaced by zero blocks to achieve a lifting.

For  $i \in \{2, \dots, c\}$  there is a natural embedding of  $N_{i-1}/N_i \hookrightarrow (V_i/V_{i+1})^{s_i}$  via the action of  $N_{i-1}/N_i$  on  $V/V_{i+1}$ , recorded on the natural basis. The action of  $H$  on  $V_i/V_{i+1}$  induces, for each  $h \in H$ , a map

$$\tau_i(h): N_{i-1}/N_i \hookrightarrow (V_i/V_{i+1})^{s_i}.$$

Define  $\vartheta_{i-1}: H \rightarrow \mathbb{R}_{\geq 0}$  by setting

$$\vartheta_{i-1}(h) = \mu_{N_{i-1}/N_i}(\{nN_i \in N_{i-1}/N_i \mid (nN_i)\tau_i(h) \in \text{Mat}_{s_i, r_i}(\mathcal{O}_{\mathfrak{p}})\}),$$

where  $\mu_{N_{i-1}/N_i}$  denotes the right Haar measure on  $N_{i-1}/N_i$ , normalised such that the set  $\psi_{i+1}^{-1}(\psi_{i+1}(N_{i-1}/N_i) \cap \text{Mat}_{s_{i+1}}(\mathcal{O}_{\mathfrak{p}}))$  has measure 1. In our application,  $c = 3$  so that there are two functions,  $\vartheta_1$  and  $\vartheta_2$ , that should be computed.

Write  $\mu_G$ , respectively  $\mu_N$  and  $\mu_H$ , for the right Haar measure on  $G$ , respectively  $N$  and  $H$ , normalised such that  $\mu_G(\mathbf{G}(\mathcal{O}_{\mathfrak{p}})) = 1$ , respectively  $\mu_N(\mathbf{N}(\mathcal{O}_{\mathfrak{p}})) = 1$  and  $\mu_H(\mathbf{H}(\mathcal{O}_{\mathfrak{p}})) = 1$ . From  $G = N \rtimes H$  one deduces that  $\mu_G = \prod_{i=2}^c \mu_{N_{i-1}/N_i} \cdot \mu_H$ . Writing  $\vartheta(h) = \mu_N(\{n \in N \mid nh \in G^+\})$  for  $h \in H$ , we can now state the following result of du Sautoy and Lubotzky.

**Theorem 3.1** ([6, Theorem 2.2]). *In the set-up described above, the various assumptions, including the ‘lifting condition’, guarantee that  $\vartheta(h) = \prod_{i=1}^{c-1} \vartheta_i(h)$  for  $h \in H$ , and consequently,*

$$\mathcal{Z}_{\mathbf{G}, \mathfrak{p}}(s) = \mathcal{Z}_{\mathbf{H}, \mathfrak{p}, \vartheta}(s), \quad \text{where} \quad \mathcal{Z}_{\mathbf{H}, \mathfrak{p}, \vartheta}(s) = \int_{H^+} |\det(h)|_{\mathfrak{p}}^s \prod_{i=1}^{c-1} \vartheta_i(h) d\mu_H(h).$$

Next we recall the machinery developed by Igusa [12] as well as du Sautoy and Lubotzky [6] for utilising a  $\mathfrak{p}$ -adic Bruhat decomposition in order to compute integrals over reductive  $\mathfrak{p}$ -adic groups; a useful reference for practical purposes is [2].

We assume that the reductive  $\mathfrak{p}$ -adic group  $H$  is split over  $\mathcal{K}_{\mathfrak{p}}$  and fix a maximal  $\mathcal{K}_{\mathfrak{p}}$ -split torus  $T = T(\mathcal{K}_{\mathfrak{p}})$  in  $H$ . In our application,  $\mathbf{H} \cong_{\mathbb{Z}} \text{GL}_n \times \text{GL}_1$  is clearly split and we take the subgroup of diagonal matrices for  $T$ . Elements of  $T$  act by conjugation on minimal closed unipotent subgroups of  $\mathbf{H}$  which correspond to elements of the root system  $\Phi$  associated to  $\mathbf{H}$ . Under this action, the (finite) Weyl group corresponds to  $W = N_{\mathbf{H}}(T)/T$ ; in our application,  $W \cong \text{Sym}(n)$ . We suppress here some necessary requirements of ‘good reduction modulo  $\mathfrak{p}$ ’ in identifying  $T$  with  $\mathbf{G}_{\mathfrak{m}}^{\dim T}$  and identifying each root subgroup with the additive group  $\mathbf{G}_a$ ; these technical requirements hold trivially in our application. Roots of  $\mathbf{H}$  relative to  $T$  are identified with elements of  $\text{Hom}(T, \mathbf{G}_{\mathfrak{m}})$  via the conjugation action of  $N_{\mathbf{H}}(T)/T$  on root subgroups. We choose a set of simple roots  $\alpha_1, \dots, \alpha_l$  which define the positive roots  $\Phi^+$ ; in our application,

$l = n - 1$ . We denote by  $\Xi = \text{Hom}(\mathbf{G}_m, \mathbf{T})$  the set of cocharacters of  $\mathbf{T}$ , and we recall the natural pairing between characters and cocharacters,

$$\text{Hom}(\mathbf{T}, \mathbf{G}_m) \times \text{Hom}(\mathbf{G}_m, \mathbf{T}) \rightarrow \mathbb{Z}, \quad (\beta, \xi) \mapsto \langle \beta, \xi \rangle,$$

defined by  $\beta(\xi(\tau)) = \tau^{\langle \beta, \xi \rangle}$  for all  $\tau \in \mathbf{G}_m$ .

We refer to [6, p. 75] for a description of Iwahori subgroup  $\mathcal{B}$  of the reductive  $\mathfrak{p}$ -adic group  $H$ . Recalling that  $\pi$  denotes a uniformising parameter for  $\mathcal{O}_{\mathfrak{p}}$ , we state the  $\mathfrak{p}$ -adic Bruhat decompositions of Iwahori and Matsumoto [13]:

$$H = \coprod_{\substack{w \in W \\ \xi \in \Xi}} \mathcal{B} w \xi(\pi) \mathcal{B} \quad \text{and} \quad \mathbf{H}(\mathcal{O}_{\mathfrak{p}}) = \coprod_{w \in W} \mathcal{B} w \mathcal{B}$$

where elements of  $W$  are suitably interpreted as representatives in  $N_H(T)$ . One defines  $\Xi^+ = \{\xi \in \Xi \mid \xi(\pi) \in \mathbf{H}(\mathcal{O}_{\mathfrak{p}})\}$  and

$$w \Xi_w^+ = \{\xi \in \Xi^+ \mid \alpha_i(\xi(\pi)) \in \mathcal{O}_{\mathfrak{p}} \text{ for all } i \in \{1, \dots, l\}, \\ \text{and } \alpha_i(\xi(\pi)) \in \pi \mathcal{O}_{\mathfrak{p}} \text{ whenever } \alpha_i \in w(\Phi^-)\},$$

where  $\Phi^-$  denotes the set of negative roots. As in [2, Section 5.2], it will turn out to be convenient to judiciously choose a basis for  $\text{Hom}(\mathbf{T}, \mathbf{G}_m)$  consisting of simple roots  $\alpha_1, \dots, \alpha_l$  and inverses  $\omega_i^{-1}$  of the dominant weights for the contragredient representations of  $H$  acting on  $V_i/V_{i+1}$ . Using the dual basis for  $\Xi = \text{Hom}(\mathbf{G}_m, \mathbf{T})$ , we thereby obtain an explicit description of the sets  $w \Xi_w^+$ .

Utilising symmetries in the affine Weyl group and the fact that the functions  $|\det|_{\mathfrak{p}}$ ,  $\vartheta$  are constant on double cosets of the Iwahori subgroup  $\mathcal{B} \leq \mathbf{H}(\mathcal{O}_{\mathfrak{p}})$ , the first author proved the following proposition, generalising results of du Sautoy and Lubotzky [6] and Igusa [12].

**Proposition 3.2** ([2, Proposition 4.2]). *Suppose that  $\mathbf{H}$  has a  $\mathcal{K}$ -split torus  $\mathbf{T}$ . Under good reduction assumptions, which are satisfied for almost all primes  $\mathfrak{p}$ ,*

$$\mathcal{Z}_{\mathbf{H}, \mathfrak{p}, \vartheta}(s) = \sum_{w \in W} q^{-\ell(w)} \sum_{\xi \in w \Xi_w^+} q^{\langle \Pi_{\beta \in \Phi^+} \beta, \xi \rangle} |\det(\xi(\pi))|_{\mathfrak{p}}^s \vartheta(\xi(\pi))$$

where  $\ell$  denotes the standard length function on the Weyl group  $W$ .

In our application, we have, in fact, good reduction for all primes.

**Remark 3.3.** Although we have not yet calculated the pro-isomorphic zeta functions of  $\mathcal{O}_{\mathfrak{p}} \otimes_{\mathbb{Z}} L_{m,n}$ , we are already able to deduce that the former will satisfy functional equations. We do so by invoking [2, Theorem 1.1]. Indeed, one can define a faithful representation  $\varrho: \mathbf{GL}_n \times \mathbf{GL}_1 \rightarrow \mathbf{H}$  given by the description of the algebraic automorphism groups in Theorem 2.7. This representation preserves integrality and preserves the Haar measure. The conditions for Theorem 1.1 in [2] are that the group  $\mathbf{H}$  should split over  $\mathcal{K}$ , that the ‘lifting condition’ should hold and that the number  $r$  of irreducible components of the representation should equal the dimension  $d$  of a maximal central torus of  $\mathbf{GL}_n \times \mathbf{GL}_1$ . The first two conditions hold. The third condition does not hold as we have  $r = 3$ ,  $d = 2$ . However, since the third block corresponds to the action on the centre and every element of the centre is obtained as a commutator of

suitable generators, it is possible to express the dominant weight of the contragredient representation for the last block  $A$  as a positive linear combination of weights occurring in the first two blocks  $\lambda\varrho_1(A), \lambda^{-1}\varrho_2(A)$  (and hence as a positive linear combination of the dominant weights for the first two blocks, together with simple roots in the root system associated to  $\mathbf{H}$  relative to a choice of maximal torus). In this way, all the requirements used to prove a functional equation in Section 5.2 of [2] are fulfilled, and the argument there implies the existence of functional equations in our situation.

#### 4. COMPUTATION OF THE LOCAL ZETA FUNCTIONS

In order to establish Theorem 1.4 it remains to carry out the detailed computations, following the framework described in Section 3. Let  $m, n \in \mathbb{N}$  with  $n \geq 2$ , and put  $d = r_1 + r_2 + r_3$ , where

$$r_1 = |\mathbf{E}| = \binom{m+n-2}{n-1}, \quad r_2 = |\mathbf{F}| = \binom{m+n-1}{n-1}, \quad r_3 = n.$$

We consider the affine group scheme  $\mathbf{G} = \mathbf{N} \rtimes \mathbf{H} \subseteq \mathbf{GL}_d$  described in Theorem 2.7. Putting  $\mathbf{H}_0 = \mathbf{GL}_n \times \mathbf{GL}_1$ , the group  $\mathbf{H}$  is the image of the faithful representation

$$(4.1) \quad \varrho: \mathbf{H}_0 \hookrightarrow \mathbf{GL}_d, \quad (A, \lambda) \mapsto \begin{pmatrix} \lambda\varrho_1(A) & & \\ & \lambda^{-1}\varrho_2(A) & \\ & & A \end{pmatrix},$$

where  $\varrho_1, \varrho_2$  are the symmetric power representations of  $\mathbf{GL}_n$  described in the proof of Lemma 2.6. We denote by  $\mathbf{T}_0$  the maximal torus of  $\mathbf{H}_0$  consisting of diagonal matrices. Via this choice, the root system  $\Phi$  of  $\mathbf{H}_0$  has simple roots  $\alpha_1, \dots, \alpha_{n-1}$  given by

$$\alpha_i(t) = a_i/a_{i+1} \quad \text{for } 1 \leq i \leq n-1 \text{ and } t = (\text{diag}(a_1, \dots, a_n), \lambda) \in \mathbf{T}_0.$$

The corresponding Weyl group is  $W \cong \text{Sym}(n)$ .

The representation  $\varrho$  in (4.1) has three irreducible components, of dimensions  $r_1, r_2$  and  $r_3$  respectively. The inverses of the dominant weights for their contragredient representations take the following values at  $t = (\text{diag}(a_1, \dots, a_n), \lambda) \in \mathbf{T}_0$ :

$$\omega_1^{-1}(t) = \lambda a_1^{-(m-1)}, \quad \omega_2^{-1}(t) = \lambda^{-1} a_n^m, \quad \omega_3^{-1}(t) = a_n.$$

In particular, we observe that  $\omega_3^{-1} = \omega_1^{-1} \omega_2^{-1} (\alpha_1 \alpha_2 \cdots \alpha_{n-1})^{m-1}$ . Furthermore, the identities

$$a_i = (\omega_3^{-1} \alpha_{n-1} \alpha_{n-2} \cdots \alpha_i)(t), \quad \text{for } 1 \leq i \leq n, \quad \text{and} \quad \lambda = \omega_1^{-1}(t) a_1^{m-1}$$

show that

$$\alpha_1, \dots, \alpha_{n-1} \quad \text{and} \quad \alpha_0 := \omega_1^{-1}, \quad \alpha_n := \omega_2^{-1}$$

form a basis for  $\text{Hom}(\mathbf{T}_0, \mathbf{G}_m)$ . We define  $\xi_0, \dots, \xi_n \in \text{Hom}(\mathbf{G}_m, \mathbf{T}_0)$  to be the dual basis. This implies, for  $\tau \in \mathbf{G}_m$ ,

$$\xi_i(\tau) = \left( \text{diag}(\underbrace{\tau^m, \dots, \tau^m}_{i \text{ entries}}, \underbrace{\tau^{m-1}, \dots, \tau^{m-1}}_{n-i \text{ entries}}), \tau^{(m-1)m} \right), \quad \text{for } 1 \leq i \leq n-1,$$

$$\xi_0(\tau) = \left( \text{diag}(\tau, \dots, \tau), \tau^m \right), \quad \text{and} \quad \xi_n(\tau) = \left( \text{diag}(\tau, \dots, \tau), \tau^{m-1} \right).$$

As before, consider a number field  $\mathcal{K}$ . For a finite prime  $\mathfrak{p}$ , let  $\mathcal{K}_{\mathfrak{p}}$  denote the completion at  $\mathfrak{p}$ , and let  $\mathcal{O}_{\mathfrak{p}}$  denote the valuation ring of  $\mathcal{K}_{\mathfrak{p}}$ . Fix a uniformising parameter  $\pi$  for  $\mathcal{O}_{\mathfrak{p}}$ , and let  $q$  denote the size of the residue field  $\mathcal{O}_{\mathfrak{p}}/\pi\mathcal{O}_{\mathfrak{p}}$ . We put

$$H_0 = H_0(\mathcal{K}_{\mathfrak{p}}) \quad \text{and} \quad T_0 = T_0(\mathcal{K}_{\mathfrak{p}}).$$

Writing  $\xi = \prod_{i=0}^n \xi_i^{e_i}$ , we obtain

$$\xi(\pi) = (A, \lambda) = (\text{diag}(a_1, \dots, a_n), \lambda) \in T_0,$$

where

$$(4.2) \quad \begin{aligned} a_i &= \pi^{(m-1)(\sum_{l=1}^{i-1} e_l) + m(\sum_{l=i}^{n-1} e_l) + (e_0 + e_n)}, \quad \text{for } 1 \leq i \leq n, \\ \lambda &= \pi^{m(m-1)(\sum_{l=1}^{n-1} e_l) + me_0 + (m-1)e_n}. \end{aligned}$$

We observe that, for non-negative exponents  $e_i$ ,  $i \in \{1, \dots, n-1\}$ , the  $\mathfrak{p}$ -adic absolute values of the first  $n$  diagonal entries of  $\xi(\pi)$  are increasing:  $|a_1|_{\mathfrak{p}} \leq \dots \leq |a_n|_{\mathfrak{p}}$ . Furthermore, we obtain

$$\begin{aligned} \det(\varrho(\xi(\pi))) &= \lambda^{r_1} \det(\varrho_1(A)) \cdot \lambda^{-r_2} \det(\varrho_2(A)) \cdot \det(A) \\ &= \lambda^{r_1} (\det(A))^{-(m-1)r_1/n} \cdot \lambda^{-r_2} (\det(A))^{mr_2/n} \cdot \det(A) \\ &= \lambda^{-\binom{m+n-2}{n-2}} (\det(A))^{1+\binom{m+n-2}{n-1}}, \end{aligned}$$

where

$$\det(A) = \prod_{i=1}^n a_i = \pi^{n(m-1)(\sum_{l=1}^{n-1} e_l) + (\sum_{l=1}^{n-1} l e_l) + n(e_0 + e_n)}.$$

Altogether we obtain

$$\begin{aligned} \det(\varrho(\xi(\pi))) &= \pi^{-\binom{m+n-2}{n-2} [m(m-1)(\sum_{l=1}^{n-1} e_l) + me_0 + (m-1)e_n]} \\ &\quad \cdot \pi^{(1+\binom{m+n-2}{n-1}) [n(m-1)(\sum_{l=1}^{n-1} e_l) + (\sum_{l=1}^{n-1} l e_l) + n(e_0 + e_n)]}, \end{aligned}$$

where the coefficient of  $e_i$  for  $1 \leq i \leq n-1$  in the exponent is

$$(4.3) \quad \text{coeff. of } e_i: \quad -m(m-1)\binom{m+n-2}{m} + (1 + \binom{m+n-2}{m-1})((m-1)n + i)$$

and the coefficients of  $e_0$  and  $e_n$  are

$$(4.4) \quad \text{coeff. of } e_0: \quad n(1 + \binom{m+n-2}{m-1}) - m\binom{m+n-2}{m} = \binom{m+n-2}{m-1} + n,$$

$$(4.5) \quad \text{coeff. of } e_n: \quad n(1 + \binom{m+n-2}{m-1}) - (m-1)\binom{m+n-2}{m} = \binom{m+n-1}{m} + n.$$

Next we work out the term  $\vartheta(\varrho(\xi(\pi)))$  which appears in Proposition 3.2, using  $\vartheta_1, \vartheta_2$ . As we already noticed, by (4.2), it is enough for our purposes to consider elements

$$(4.6) \quad t = (A, \lambda) = (\text{diag}(a_1, \dots, a_n), \lambda) \in T_0 \quad \text{with } |a_1|_{\mathfrak{p}} \leq \dots \leq |a_n|_{\mathfrak{p}}.$$

Using this restriction and writing  $v_{\mathfrak{p}}(\cdot) = -\log_q |\cdot|_{\mathfrak{p}}$ , we obtain from the description of the unipotent radical in Theorem 2.7,

$$(4.7) \quad \log_q(\vartheta_1(\varrho(t))) = \sum_{k=1}^n \binom{m+k-2}{k-1} \sum_{\mathbf{f} \in \mathbf{F}(k)} v_{\mathfrak{p}} \left( \lambda^{-1} \prod_{i=k}^n a_i^{f_i} \right),$$

where

$$\mathbf{F}(k) = \{\mathbf{f} \mid \mathbf{f} = (f_1, \dots, f_n) \in \mathbf{F} \text{ with } f_1 = \dots = f_{k-1} = 0, f_k \geq 1\}, \quad \text{for } 1 \leq k \leq n.$$

This formula can be justified as follows. Referring to the notation in Theorem 2.7, the entries of the matrix  $C = (b_{\mathbf{e}+\mathbf{f}})_{(\mathbf{e}, \mathbf{f}) \in \mathbf{E} \times \mathbf{F}}$  are indexed by the elements of

$$\mathbf{G} = \{\mathbf{g} \mid \mathbf{g} = (g_1, \dots, g_n) \in \mathbb{N}_0^n \text{ with } g_1 + \dots + g_n = 2m - 1\}.$$

For  $\mathbf{g} = (g_1, \dots, g_n) \in \mathbf{G}$ , the value  $b_{\mathbf{g}}$  occurs in the column labeled by  $\mathbf{f} = (f_1, \dots, f_n) \in \mathbf{F}$  if and only if  $\mathbf{f} \leq \mathbf{g}$ , i.e.,  $f_1 \leq g_1, \dots, f_n \leq g_n$ . The  $(\mathbf{f}, \mathbf{f})$ -entry of the diagonal matrix  $\lambda^{-1} \varrho_2(A)$  is equal to  $\lambda^{-1} \prod_{i=1}^n a_i^{f_i}$ . This yields the condition

$$v_{\mathbf{p}}(b_{\mathbf{g}}) \geq \max \left\{ -v_{\mathbf{p}} \left( \lambda^{-1} \prod_{i=1}^n a_i^{f_i} \right) \mid \mathbf{f} \in \mathbf{F} \text{ with } \mathbf{f} \leq \mathbf{g} \right\}.$$

We observe that the maximum on the right-hand side is attained for  $\mathbf{f} \leq \mathbf{g}$  such that  $\mathbf{f}$  is minimal with respect to the lexicographical (non-cyclical) ordering on  $\mathbf{F}$ ; this follows from the restriction (4.6). We arrange the summation over  $\mathbf{F} = \mathbf{F}(1) \sqcup \dots \sqcup \mathbf{F}(n)$  rather than  $\mathbf{G}$ . For  $1 \leq k \leq n$  and  $\mathbf{f} \in \mathbf{F}(k)$ ,

$$\# \left\{ \mathbf{g} \in \mathbf{G} \mid \mathbf{f} \leq \mathbf{g} \text{ and } \neg \exists \mathbf{f}' \in \mathbf{F}(k) \cup \dots \cup \mathbf{F}(n) : \mathbf{f}' \leq \mathbf{g} \wedge f'_k < f_k \right\}$$

is equal to the number of ways of distributing  $m - 1$  increments among the first  $k$  coordinates, giving  $\binom{m-1+k-1}{k-1} = \binom{m+k-2}{k-1}$ . In this way we obtain the summands

$$\binom{m+k-2}{k-1} v_{\mathbf{p}} \left( \lambda^{-1} \prod_{i=k}^n a_i^{f_i} \right), \quad \text{for each } \mathbf{f} \in \mathbf{F}(k),$$

in the formula (4.7).

Observe that  $|\mathbf{F}(k)| = \binom{m+(n-k+1)-1}{(n-k+1)-1} - \binom{m+(n-k)-1}{(n-k)-1} = \binom{m+n-k-1}{n-k}$ , for  $1 \leq k \leq n$ . Writing  $N(n, m) = \sum_{\mathbf{f} \in \mathbf{F}} f_1 = \sum_{\mathbf{f} \in \mathbf{F}(1)} f_1 = \frac{m}{n} \binom{m+n-1}{n-1} = \binom{m+n-1}{n}$ , we deduce from (4.7) that

$$\begin{aligned} \log_q(\vartheta_1(\varrho(t))) &= \sum_{k=1}^n \binom{m+k-2}{k-1} \sum_{\mathbf{f} \in \mathbf{F}(k)} v_{\mathbf{p}} \left( \lambda^{-1} \prod_{i=k}^n a_i^{f_i} \right) \\ &= \sum_{k=1}^n \binom{m+k-2}{k-1} |\mathbf{F}(k)| v_{\mathbf{p}}(\lambda^{-1}) \\ &\quad + \sum_{i=1}^n \left( \sum_{k=1}^i \binom{m+k-2}{k-1} \sum_{\mathbf{f} \in \mathbf{F}(k)} f_i \right) v_{\mathbf{p}}(a_i) \\ &= \sum_{k=1}^n \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k} v_{\mathbf{p}}(\lambda^{-1}) \\ &\quad + \sum_{i=1}^n \left( \binom{m+i-2}{i-1} N(n-i+1, m) \right. \\ &\quad \left. + \sum_{k=1}^{i-1} \binom{m+k-2}{k-1} (N(n-k+1, m) - N(n-k, m)) \right) v_{\mathbf{p}}(a_i) \\ &= \binom{2m+n-2}{n-1} v_{\mathbf{p}}(\lambda^{-1}) \\ &\quad + \sum_{i=1}^n \left( \binom{m+i-2}{i-1} \binom{m+n-i-1}{n-i} + \sum_{k=1}^i \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} \right) v_{\mathbf{p}}(a_i). \end{aligned}$$

Using (4.2) and writing

$$R(m, n) = \sum_{i=1}^n \left( \binom{m+i-2}{i-1} \binom{m+n-i-1}{n-i} + \sum_{k=1}^i \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} \right) - m \binom{2m+n-2}{n-1},$$

we deduce that

$$\begin{aligned}
& \log_q(\vartheta_1(\varrho(\xi(\pi)))) \\
&= -\binom{2m+n-2}{n-1} \left( m(m-1) \left( \sum_{l=1}^{n-1} e_l \right) + me_0 + (m-1)e_n \right) \\
&+ \sum_{i=1}^n \left( \binom{m+i-2}{i-1} \binom{m+n-i-1}{n-i} + \sum_{k=1}^i \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} \right) \\
&\quad \cdot \left( (m-1) \left( \sum_{l=1}^{n-1} e_l \right) + \left( \sum_{l=i}^{n-1} e_l \right) + (e_0 + e_n) \right) \\
&= \sum_{j=1}^{n-1} \left( (m-1)R(m, n) + \sum_{i=1}^j \left( \binom{m+i-2}{i-1} \binom{m+n-i-1}{n-i} \right. \right. \\
&\quad \left. \left. + \sum_{k=1}^i \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} \right) \right) e_j + R(m, n)e_0 + \left( R(m, n) + \binom{2m+n-2}{n-1} \right) e_n.
\end{aligned}$$

We observe that, in fact,

$$\begin{aligned}
R(m, n) &= \sum_{i=1}^n \binom{m+i-2}{i-1} \binom{m+n-i-1}{n-i} + \sum_{k=1}^n \sum_{i=k}^n \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} - m \binom{2m+n-2}{n-1} \\
&= \binom{2m+n-2}{n-1} + \sum_{k=1}^n (n-k+1) \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} - m \binom{2m+n-2}{n-1} \\
&= (m-1) \sum_{k=1}^n \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k} - (m-1) \binom{2m+n-2}{n-1} \\
&= 0.
\end{aligned}$$

Thus we obtain

$$\log_q(\vartheta_1(\varrho(\xi(\pi)))) = \sum_{i=1}^{n-1} C_i(m, n)e_i + \binom{2m+n-2}{n-1}e_n,$$

where

$$\begin{aligned}
C_i(m, n) &= \sum_{j=1}^i \left( \binom{m+j-2}{j-1} \binom{m+n-j-1}{n-j} + \sum_{k=1}^j \binom{m+k-2}{k-1} \binom{m+n-k-1}{n-k+1} \right) \\
&= \sum_{j=1}^i \binom{m+j-2}{m-1} \binom{m+n-j-1}{m-1} + \sum_{k=1}^i (i-k+1) \binom{m+k-2}{m-1} \binom{m+n-k-1}{m-2} \\
&= \sum_{j=1}^i \left( 1 + \frac{(m-1)(i-j+1)}{n-j+1} \right) \binom{m+j-2}{m-1} \binom{m+n-j-1}{m-1}.
\end{aligned}$$

Hence we arrive at

$$(4.8) \quad \log_q(\vartheta_1(\varrho(\xi(\pi)))) = \sum_{i=1}^{n-1} \left( \sum_{j=1}^i \left( 1 + \frac{(m-1)(i-j+1)}{n-j+1} \right) \binom{m+j-2}{m-1} \binom{m+n-j-1}{m-1} \right) e_i + \binom{2m+n-2}{n-1} e_n.$$

This finishes our computation of  $\vartheta_1(\varrho(\xi(\pi)))$ . It is much easier to determine  $\vartheta_2(\varrho(\xi(\pi)))$ . Indeed, from the description of the unipotent radical in Theorem 2.7 we observe that, for  $t = (\text{diag}(a_1, \dots, a_n), \lambda) \in T_0$ ,

$$\log_q(\vartheta_2(\varrho(t))) = (r_1 + r_2) \sum_{i=1}^n v_{\mathbf{p}}(a_i) = \left( \binom{m+n-2}{n-1} + \binom{m+n-1}{n-1} \right) \sum_{i=1}^n v_{\mathbf{p}}(a_i),$$

and hence, using (4.2),

$$\begin{aligned}
(4.9) \quad \log_q(\vartheta_2(\varrho(\xi(\pi)))) &= \left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right) \\
&\quad \cdot \left( ((m-1)n + l) \left( \sum_{l=1}^{n-1} e_l \right) + n(e_0 + e_n) \right).
\end{aligned}$$

We are ready to apply Proposition 3.2 to the group  $H \cong_{\mathbb{Z}} H_0$ . Writing  $\beta_0 = \prod_{\beta \in \Phi^+} \beta$  and denoting by  $\ell$  the standard length function on the Weyl group  $W \cong \text{Sym}(n)$ , we obtain

$$\begin{aligned} \mathcal{Z}_{H,p,\vartheta}(s) &= \sum_{w \in W} q^{-\ell(w)} \sum_{\xi \in w\Xi_w^+} q^{\langle \beta_0, \xi \rangle} |\det(\varrho(\xi(\pi)))|^s \vartheta_1(\varrho(\xi(\pi))) \vartheta_2(\varrho(\xi(\pi))) \\ &= \frac{\sum_{w \in W} q^{-\ell(w)} \prod_{i=1}^{n-1} X_i^{\nu_i(w)}}{\prod_{i=1}^{n-1} (1 - X_i)} \cdot \frac{1}{(1 - \tilde{X}_0)(1 - \tilde{X}_n)}, \end{aligned}$$

where

$$\nu_i(w) = \begin{cases} 1 & \text{if } \alpha_i \in w(\Phi^-), \\ 0 & \text{otherwise,} \end{cases}$$

each  $X_i$ , for  $1 \leq i \leq n-1$ , accounts for the ‘ $e_i$ -contributions’ of  $\xi = \prod_{i=0}^n \xi_i^{e_i}$  and  $\tilde{X}_0, \tilde{X}_n$  account for the contributions of  $e_0, e_n$ , as specified below.

We observe that  $\beta_0 = \prod_{i=1}^{n-1} \alpha_i^{i(n-i)}$ , hence  $\langle \beta_0, \xi \rangle = \sum_{i=1}^{n-1} i(n-i)e_i$ . Moreover, for  $1 \leq i \leq n-1$ ,

$$\begin{aligned} \log_q(X_i) &= i(n-i) + \underbrace{\left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right)}_{\text{contributions from } \vartheta_2 \text{ according to (4.9)}} ((m-1)n+i) \\ &\quad + \underbrace{\sum_{j=1}^i \left( 1 + \frac{(m-1)(i-j+1)}{n-j+1} \right) \binom{m+j-2}{m-1} \binom{m+n-j-1}{m-1}}_{\text{contributions from } \vartheta_1 \text{ according to (4.8)}} \\ &\quad - \underbrace{\left( -m(m-1) \binom{m+n-2}{m} + \left( 1 + \binom{m+n-2}{m-1} \right) ((m-1)n+i) \right)}_{\text{contributions from the determinant according to (4.3)}} s \end{aligned}$$

and

$$\begin{aligned} \log_q(\tilde{X}_0) &= n \underbrace{\left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right)}_{\text{contributions from (4.9)}} - \underbrace{\left( \binom{m+n-2}{m-1} + n \right)}_{\text{contributions from (4.4)}} s, \\ \log_q(\tilde{X}_n) &= n \underbrace{\left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} \right)}_{\text{contributions from (4.9) and (4.8)}} + \binom{2m+n-2}{2m-1} - \underbrace{\left( \binom{m+n-1}{m} + n \right)}_{\text{contributions from (4.5)}} s. \end{aligned}$$

In view of Remark 2.2, this completes the proof of Theorem 1.4.

## 5. LOCAL FUNCTIONAL EQUATIONS, ABCISSAE OF CONVERGENCE AND MEROMORPHIC CONTINUATION

In this section we analyse the explicit formulae in Theorem 1.4 to deduce Corollaries 1.5, 1.6 and 1.8.

*Proof of Corollary 1.5.* The formula we obtained in Section 4 for  $\mathcal{Z}(s) = \mathcal{Z}_{H,p,\vartheta}(s)$  is classical and was treated by Igusa [12]; also see [14, Section 2]. We follow the adapted approach given by du Sautoy and Lubotzky [6, p. 82]. Let  $w_0$  denote the unique element of maximal length  $\ell(w_0) = \binom{n}{2}$  in the Weyl group  $W \cong \text{Sym}(n)$ . It is well-known that



$\ell(w) + \ell(w w_0) = |\Phi^+|$  and  $w_0(\Phi^-) = \Phi^+$  for all  $w \in W$ ; see, for instance, [11, Section 1.8]. The latter implies that  $\nu_i(w) = 1 - \nu_i(w w_0)$  for  $w \in W$  and  $i \in \{1, \dots, n-1\}$ . Hence

$$\begin{aligned} \mathcal{Z}(s)|_{q \rightarrow q^{-1}} &= \frac{\sum_{w \in W} q^{\ell(w)} (-1)^{n-1} \prod_{i=1}^{n-1} X_i^{1-\nu_i(w)}}{\prod_{i=1}^{n-1} (1 - X_i)} \cdot \frac{\tilde{X}_0 \tilde{X}_n}{(1 - \tilde{X}_0)(1 - \tilde{X}_n)} \\ &= (-1)^{n-1} q^{|\Phi^+|} \frac{\sum_{w \in W} q^{-\ell(w w_0)} \prod_{i=1}^{n-1} X_i^{\nu_i(w w_0)}}{\prod_{i=1}^{n-1} (1 - X_i)} \cdot \frac{\tilde{X}_0 \tilde{X}_n}{(1 - \tilde{X}_0)(1 - \tilde{X}_n)} \\ &= (-1)^{n-1} q^{|\Phi^+|} \tilde{X}_0 \tilde{X}_n \mathcal{Z}(s). \end{aligned}$$

Noting that  $|\Phi^+| = \binom{n}{2}$ , we obtain a symmetry factor of  $q^{a+bs}$ , where the parameters  $a$  and  $b$  are given by (1.4). The special case  $\mathcal{K} = \mathbb{Q}$  yields the functional equation for the local factors  $\zeta_{\Delta_{m,n,p}}^\wedge(s)$ .  $\square$

In preparation for the proof of Corollary 1.6 we derive two lemmata.

**Lemma 5.1.** *Let  $A_i, B_i$ , for  $0 \leq i \leq n$ , and  $\tilde{A}_j, \tilde{B}_j$ , for  $j \in \{0, n\}$  be as in Theorem 1.4. Suppose that  $m \geq 2$ . Then*

$$\max \left\{ \frac{A_i + 1}{B_i} \mid 1 \leq i \leq n-1 \right\} < \max \left\{ \frac{\tilde{A}_0 + 1}{\tilde{B}_0}, \frac{\tilde{A}_n + 1}{\tilde{B}_n} \right\}.$$

*Proof.* We observe that

$$\frac{\tilde{A}_0 + 1}{\tilde{B}_0} = \frac{A_0 + (m-1)}{B_0} \geq \frac{A_0 + 1}{B_0}$$

and similarly  $(\tilde{A}_n + 1)/\tilde{B}_n \geq (A_n + 1)/B_n$ . Consequently it suffices to show that

$$\max \left\{ \frac{A_i + 1}{B_i} \mid 1 \leq i \leq n-1 \right\} < \max \left\{ \frac{A_0 + 1}{B_0}, \frac{A_n + 1}{B_n} \right\}.$$

Observe that for any positive numbers  $a, b, x, y$  we have

$$(5.1) \quad \frac{x}{y} \geq \frac{a+x}{b+y} \geq \frac{a}{b} \iff \frac{x}{y} \geq \frac{a}{b},$$

and similarly with strict inequalities. Writing  $x_i = (A_i + 1) - (A_{i-1} + 1) = A_i - A_{i-1}$  and  $y_i = B_i - B_{i-1}$  for  $1 \leq i \leq n$ , we deduce that it suffices to prove that

$$0 < x_1/y_1 < \dots < x_n/y_n.$$

For then (5.1) implies that  $(A_i + 1)/B_i$ ,  $0 \leq i \leq n$ , forms a convex sequence and takes its maximum at  $i = 0$  or  $i = n$ .

In fact,  $y_i = 1 + \binom{m+n-2}{m-1}$  is constant and, for  $1 \leq i \leq n$ ,

$$\begin{aligned} x_i &= \left( \binom{m+n-2}{m-1} + \binom{m+n-1}{m} + n + 1 \right) - 2i \\ &\quad + \binom{m+i-2}{m-1} \binom{m+n-i-1}{m-1} + \sum_{j=1}^i \frac{m-1}{n-j+1} \binom{m+j-2}{m-1} \binom{m+n-j-1}{m-1}. \end{aligned}$$

Clearly,  $x_1 \geq n - 1 > 0$ , and it remains to check that  $d_i = x_i - x_{i-1}$  is positive for  $2 \leq i \leq n$ . Indeed, we obtain

$$\begin{aligned}
d_i &= -2 + \left( \frac{m+n-i}{n-i+1} - \frac{(i-1)(m+n-i)}{(m+i-2)(n-i+1)} \right) \binom{m+i-2}{m-1} \binom{m+n-i-1}{m-1} \\
&= -2 + \frac{(m+n-i)(m-1)}{(n-i+1)(m+i-2)} \frac{(m+i-2)!}{(m-1)!(i-1)!} \frac{(m+n-i-1)!}{(m-1)!(n-i)!} \\
&= -2 + \frac{(m+i-3)(m+i-4) \cdots i}{(m-2)!} \frac{(m+n-i)(m+n-i-1) \cdots (n-i+2)}{(m-1)!} \\
&\geq -2 + i(n-i+2) \\
&> 0.
\end{aligned}$$

□

**Lemma 5.2.** *Let  $\tilde{A}_j, \tilde{B}_j$ , for  $j \in \{0, n\}$  be as in Theorem 1.4, and let  $\mathcal{C}$  be as in Corollary 1.6. Suppose that  $m \geq 2$ . Then*

$$\frac{\tilde{A}_0 + 1}{\tilde{B}_0} < \frac{\tilde{A}_n + 1}{\tilde{B}_n} \iff \frac{\tilde{A}_0 + 1}{\tilde{B}_0} \leq \frac{\tilde{A}_n + 1}{\tilde{B}_n} \iff (m, n) \notin \mathcal{C}.$$

*Proof.* Setting

$$F(m, n) = \frac{\binom{2m+n-2}{2m-1}}{\binom{m+n-2}{m}} - \frac{\binom{m+n-2}{m-1} \left( \frac{n(n-1)}{m} + 2n \right) + 1}{\binom{m+n-2}{m-1} + n},$$

we observe from (5.1) and the formulae in Theorem 1.4 that

$$\frac{\tilde{A}_0 + 1}{\tilde{B}_0} \leq \frac{\tilde{A}_n + 1}{\tilde{B}_n} \iff \frac{\tilde{A}_n - \tilde{A}_0}{\tilde{B}_n - \tilde{B}_0} \geq \frac{\tilde{A}_0 + 1}{\tilde{B}_0} \iff F(m, n) \geq 0,$$

and similarly with strict inequalities. A standard calculation shows that

$$F(m, n) = \frac{f_m(n)}{\frac{(2m-1)!}{(m-1)!} (n-1)n \left( \prod_{i=1}^{m-2} (n+i) + (m-1)! \right)},$$

where

$$\begin{aligned}
f_m &= mt \left( \prod_{i=m-1}^{2m-2} (t+i) \right) \left( \prod_{i=1}^{m-2} (t+i) + (m-1)! \right) \\
&\quad - (t-1) \left( \prod_{i=m+1}^{2m-1} i \right) \left( t^2(t+2m-1) \prod_{i=1}^{m-2} (t+i) + m! \right) \in \mathbb{Z}[t].
\end{aligned}$$

We need to prove that  $f_m(n) \geq 0$  if and only if  $(m, n) \notin \mathcal{C}$ , and that in these cases even the strict inequality  $f_m(n) > 0$  holds. For  $1 \leq m \leq 6$ , we compute  $f_m$  explicitly and a routine analysis yields the desired result; see Table 1.

To conclude the proof it suffices to show that, for  $m \geq 7$ , the polynomial  $f_m$  has non-negative coefficients. For  $7 \leq m \leq 29$  this can be checked directly. Now suppose

TABLE 1. The polynomials  $f_m$  for  $1 \leq m \leq 6$ 

$m$	$f_m$	comments
2	$-3t^4 - 2t^3 + 21t^2 + 2t + 6$	$f_2(2) = 30, f_2(3) = -96,$ $f'_2(x) < 0$ for $x \geq 2$
3	$-17t^5 - 64t^4 + 179t^3 + 406t^2 + 96t + 120$	$f_3(2) = 1800, f_3(3) = -420,$ $f'_3(x) < 0$ for $x \geq 3$
4	$4t^7 - 126t^6 - 1166t^5 + 642t^4 + 11242t^3$ $+18204t^2 + 6480t + 5040$	$f_4(n) > 0$ for $n \in \{2, 3, 39\},$ $f_4(n) < 0$ for $n \in \{4, 5, \dots, 38\},$ $f'_4(x) > 0$ for $x \geq 34$
5	$5t^9 + 180t^8 - 294t^7 - 19536t^6 - 35355t^5$ $+258060t^4 + 993244t^3 + 1424496t^2$ $+645120t + 362880$	$f_5(n) > 0$ for $n \in \{2, 3, 4, 10\},$ $f_5(n) < 0$ for $n \in \{5, 6, 7, 8, 9\},$ $f'_5(x) > 0$ for $x \geq 9$
6	$6t^{11} + 330t^{10} + 7920t^9 + 53460t^8 - 161442t^7$ $-1429830t^6 + 5025180t^5 + 48636840t^4$ $+125765136t^3 + 170467200t^2 + 90720000t$ $+39916800$	$f_6(2) > 0,$ $f'_6(x) > 0$ for $x \geq 2,$

that  $m \geq 30$ . A short calculation reveals that

$$f_m = g_m + \left( \prod_{i=0}^{m-2} (t+i) \right) h_m,$$

where

$$g_m = m! t \prod_{i=m-1}^{2m-2} (t+i) - (2m-1)! (t-1),$$

$$h_m = m \prod_{i=m-1}^{2m-2} (t+i) - \frac{(2m-1)!}{m!} (t-1)t(t+2m-1).$$

Clearly, it is enough to prove that  $g_m$  and  $h_m$  have non-negative coefficients. As for  $g_m$ , we only have to examine the coefficient of  $t$ . It is

$$(m(m-1) - (2m-1)) (2m-2)! = (m^2 - 3m + 1)(2m-2)! \geq 0$$

for  $m \geq 3$ . For  $h_m$ , we need to examine the coefficients of  $t^3$  and  $t^2$ . The coefficient of  $t^3$  is non-negative if and only if

$$m \sum_{\substack{a,b,c \text{ with} \\ m-1 \leq a < b < c \leq 2m-2}} \prod_{\substack{m-1 \leq i \leq 2m-2 \\ \text{and } i \neq a,b,c}} i \geq \prod_{i=m+1}^{2m-1} i.$$

Here, the left-hand side is at least

$$m \binom{m}{3} \prod_{i=m-1}^{2m-5} i = \frac{m^3(m-1)^2(m-2)}{6} \prod_{i=m+1}^{2m-5} i$$

and it suffices to observe that

$$\frac{1}{6}m^3(m-1)^2(m-2) \geq (2m-1)(2m-2)(2m-3)(2m-4)$$

for  $m \geq 10$ . Similarly, the coefficient of  $t^2$  in  $h_m$  is non-negative if and only if

$$m \sum_{\substack{a,b \text{ with} \\ m-1 \leq a < b \leq 2m-2}} \prod_{\substack{m-1 \leq i \leq 2m-2 \\ \text{and } i \neq a,b}} i \geq (2m-2) \prod_{i=m+1}^{2m-1} i.$$

Here, the left-hand side is at least

$$m \binom{m}{2} \prod_{i=m-1}^{2m-4} i = \frac{m^3(m-1)^2}{2} \prod_{i=m+1}^{2m-4} i$$

and it suffices to observe that

$$\frac{1}{2}m^3(m-1)^2 \geq (2m-1)(2m-2)^2(2m-3)$$

for  $m \geq 30$ . □

*Proof of Corollary 1.6.* For  $m = 1$  it is easy to check the claim directly; see Remark 1.7. Now suppose that  $m \geq 2$ . Recall that  $\beta(m, n) = \max\{A_i/B_i \mid 1 \leq i \leq n-1\}$ , and for  $w \in W$  let

$$I(w) = \{i \mid 1 \leq i \leq n-1 \text{ and } \alpha_i \in w(\Phi^-)\}$$

denote the left descent set of  $w$ . We use the Euler product decomposition  $\zeta_{\Delta_{m,n}}^\wedge(s) = \prod_p \zeta_{\Delta_{m,n,p}}^\wedge(s)$ . Combining the formula (1.3) for the local zeta functions, obtained in Theorem 1.4, with Lemmata 5.1 and 5.2, we see that it suffices to show that the ‘product of numerators’

$$P(s) = \prod_p \left( 1 + \sum_{w \in W \setminus \{1\}} p^{-\ell(w)} \prod_{i \in I(w)} p^{A_i - B_i s} \right)$$

converges absolutely for  $\operatorname{Re}(s) > \beta(m, n)$ . Note that the strict inequalities in the lemmata imply that we get a simple pole at  $s = \alpha(m, n)$ .

Observe that an infinite product of complex numbers of the form  $\prod_p (1 + \sum_{j=1}^N z_{j,p})$  converges absolutely if and only if  $\prod_p (1 + z_{j,p})$  converges absolutely for each  $j \in \{1, \dots, N\}$ . Therefore  $P(s)$  converges absolutely if and only if

$$P_w(s) = \prod_p \left( 1 + p^{-\ell(w)} \prod_{i \in I(w)} p^{A_i - B_i s} \right) = \prod_p \left( 1 + p^{-\ell(w) + \sum_{i \in I(w)} (A_i - B_i s)} \right)$$

converges absolutely for each  $w \in W \setminus \{1\}$ . Clearly,  $P_w(s)$  converges absolutely if and only if

$$\operatorname{Re}(s) > \frac{1 - \ell(w) + \sum_{i \in I(w)} A_i}{\sum_{i \in I(w)} B_i}.$$

Noting that

$$(5.2) \quad \frac{1 - \ell(w) + \sum_{i \in I(w)} A_i}{\sum_{i \in I(w)} B_i} \leq \frac{\sum_{i \in I(w)} A_i}{\sum_{i \in I(w)} B_i} \leq \max \left\{ \frac{A_i}{B_i} \mid i \in I(w) \right\},$$

we deduce that  $P(s)$  converges absolutely for  $\operatorname{Re}(s) > \beta(m, n)$ . In fact, this method does not yield anything better, because each value  $A_i/B_i$  is indeed attained with equalities in (5.2) for a suitable simple reflection  $w$ .  $\square$

*Proof of Corollary 1.8.* We write  $\tilde{A}_n(m, n)$  and  $\tilde{B}_n(m, n)$  to indicate the dependence of  $\tilde{A}_n$  and  $\tilde{B}_n$  on the parameters  $m, n$ . From Corollary 1.6 we deduce that

$$\begin{aligned} \lim_{m \rightarrow \infty} \alpha(m, n) &= \lim_{m \rightarrow \infty} (\tilde{A}_n(m, n) + 1) / \tilde{B}_n(m, n) \\ &= \lim_{m \rightarrow \infty} \frac{n \binom{m+n-2}{n-1} + \binom{m+n-1}{n-1} + \binom{2m+n-2}{n-1} + 1}{\binom{m+n-1}{n-1} + n} \\ &= \lim_{m \rightarrow \infty} \frac{n \left( \frac{m}{m+n-1} + 1 \right) \binom{m+n-1}{n-1} + \frac{2m+n-2}{m+n-1} \cdot \frac{2m+n-3}{m+n-2} \cdots \frac{2m}{m+1} \cdot \binom{m+n-1}{n-1} + 1}{\binom{m+n-1}{n-1} + n} \\ &= 2n + 2^{n-1}. \end{aligned} \quad \square$$

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